

New Strategy of Developing a Direct Two-Step Implicit Hybrid Block Method with Generalized Two Off-Step Points for Third Order Ordinary Differential Equations

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Article Info

Volume 81

Page Number: 753 - 763

Publication Issue:

November-December 2019

Article History

Article Received: 3 January 2019

Revised: 25 March 2019

Accepted: 28 July 2019

Publication: 25 November 2019

Abstract: This paper proposes new strategy in developing a direct two-step implicit hybrid block method with generalized two off-step points for initial value problems (IVPs) of third order ordinary differential equations (ODEs). In this strategy, two off-step points are confined in the second step of two-step interval. The main continuous schemes are obtained through interpolating approximate solutions in the form of power series at three the grid points while its second derivatives are collocated at all the grid points in the interval. The analysis of the method such as order, zero stability, consistency and convergence are also discussed. It was found that the proposed method outperforms the existing methods. Hence, it can be considered as a viable alternative method to solve the third order of IVPs directly.

I. INTRODUCTION

The third order of IVPs in form

$$y''' = f(x, y, y', y''), \quad y(a) = \varphi_0, \quad y'(a) = \varphi_1, \quad y''(a) = \varphi_2, \quad x \in [a, b]. \quad (1)$$

can be solved by using direct methods such as predictor-corrector methods [1], block methods[2] – [9]and hybrid block methods[10]. However, most of existing hybrid block methods were derived using specific off-step points for solving (1) directly. In order to overcome these issues, multi-step hybrid block methods were introduced. These methods are not only capable of computing numerical solution at many points simultaneously, but they manage to overcome zero stability (see [11, 12]). Recently, Mansor et al.[13] proposed two-step hybrid block method with generalized

one off-step point to find the direct solution of the second order IVPs. As expected, this method produces better accuracy than the previous methods. However, it only considers one off-step point between two-step intervals.

This paper intends to extend their work for solving third order IVPs by considering two off-step points in the second step of two-step interval.

II. DEVELOPMENT OF THE METHOD

In this section, a two-step implicit hybrid block method with generalized two off-step points based on new strategy for solving (1) is described.

Firstly, define a two-step interval, $x \in (x_n, x_{n+2}]$ which consists of the initial step point, x_n the interior step point, x_{n+1} and the last step point,

x_{n+2} with two off-step points, x_{n+s} and x_{n+r} where $1 < s < r < 2$.

In deriving a direct two-step implicit hybrid block method using the new strategy (refer to Figure 1), we consider two off-step points i. x_{n+s} and

x_{n+r} which are located in the second step of a two-step interval. As a result, the total number of points used in deriving the proposed method is five (5).

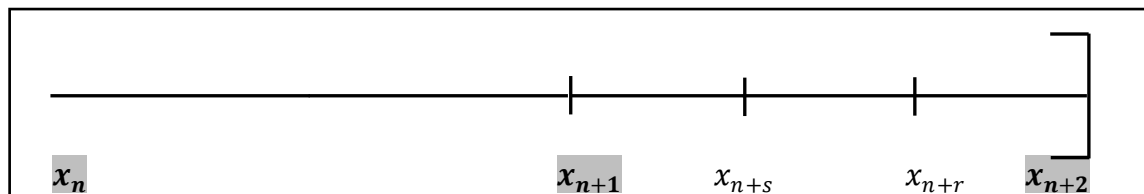


Fig 1: Two-step interval with two off-step points

Now, let the approximate solution of (1) be the power series polynomial of the following form

$$y(x) = \sum_{j=0}^{i+c-1} a_j \left(\frac{x-x_n}{h}\right)^j \quad (2)$$

where $x \in (x_n, x_{n+2}]$ for $n = 0, 2, 4, \dots, N-2$, i is the number of interpolation points which is equal to the order of differential equation, c is the number of collocation points in the interval i.e. x_{n+j} , ($j = 0, 1, s, r, 2$) and $h = x_{n+1} - x_n$ is a constant step size for the partition of interval $[a, b]$ defined by $a = x_0 < x_2 < x_4 < \dots < x_n < x_{n+2} < \dots < x_{N-2} = b$.

Differentiating (2) three time gives

$$y'''(x) = \sum_{j=3}^{i+c-1} a_j \frac{j(j-1)(j-2)}{h^3} \left(\frac{x-x_n}{h}\right)^{j-3} \quad (3)$$

Now, interpolating (2) at x_{n+j} , ($j = 0, 1, s$) and collocating (3) at all points in that interval produces eight equations which can be written in the following matrix form:

$$\frac{1}{h^3} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & s & s^2 & s^3 & s^4 & s^5 & s^6 & s^7 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 24 & 60 & 120 & 210 \\ 0 & 0 & 0 & 6 & 24s & 60s^2 & 120s^3 & 210s^4 \\ 0 & 0 & 0 & 6 & 24r & 60r^2 & 120r^3 & 210r^4 \\ 0 & 0 & 0 & 6 & 48 & 240 & 960 & 3360 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+1} \\ y_{n+s} \\ f_n \\ f_{n+1} \\ f_{n+s} \\ f_{n+r} \\ f_{n+2} \end{pmatrix} \quad (4)$$

The values of a_j 's, ($j = 0, 1, \dots, 7$) can be obtained by using Gaussian elimination and then substituted back into (2) to give a continuous implicit scheme of the form

$$y(x) = \sum_{j=0,1,s} \alpha_j(x) y_{n+j} + \sum_{j=0}^2 \beta_j(x) f_{n+j} + \sum_{j=s,r} \beta_j(x) f_{n+j} \quad (5)$$

Now, differentiating (5) twice yields

$$y^{(i)}(x) = \sum_{j=0,1,s} \alpha_j^{(i)}(x) y_{n+j} + \sum_{j=0,1,2,s,r} \beta_j^{(i)}(x) f_{n+j}, \quad i = (1)(1)(2). \quad (6)$$

Evaluating (5) at non-interpolating points, i.e. x_{n+r} and x_{n+2} , and (6) at all points i.e. x_{n+j} ,

($j = 0, 1, s, r, 2$) produces a discrete scheme and its derivatives at x_n represented by the matrix form below

$$A^{2[2]_3} Y_m^{2[2]_3} = B_1^{2[2]_3} R_1^{2[2]_3} + B_2^{2[2]_3} R_2^{2[2]_3} + B_3^{2[2]_3} R_3^{2[2]_3} + h^3 [D^{2[2]_3} R_4^{2[2]_3} + E^{2[2]_3} R_5^{2[2]_3}] \quad (7)$$

where

$$A^{2[2]_3} = \begin{pmatrix} \frac{r(s-r)}{1-s} & \frac{(r-1)r}{(1-s)s} & 1 & 0 \\ \frac{2(s-2)}{1-s} & \frac{2}{(1-s)s} & 0 & 1 \\ s & 1 & 0 & 0 \\ \frac{h(1-s)}{2} & \frac{hs^2-hs}{2} & 0 & 0 \\ \frac{h^2(s-1)}{h^2(s-1)} & \frac{h^2(1-s)s}{h^2(1-s)s} & 0 & 0 \end{pmatrix}, B_1^{2[2]_3} = \begin{pmatrix} 0 & 0 & 0 & \frac{(r-1)(r-s)}{s} \\ 0 & 0 & 0 & \frac{2}{s} - 1 \\ 0 & 0 & 0 & -\frac{1+s}{hs} \\ 0 & 0 & 0 & \frac{2}{h^2s} \end{pmatrix},$$

$$B_2^{2[2]_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B_3^{2[2]_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, D^{2[2]_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & D_{14} \\ 0 & 0 & 0 & D_{24} \\ 0 & 0 & 0 & D_{34} \\ 0 & 0 & 0 & D_{44} \end{pmatrix},$$

$$E^{2[2]_3} = \begin{pmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{21} & E_{22} & E_{23} & E_{24} \\ E_{31} & E_{32} & E_{33} & E_{34} \\ E_{41} & E_{42} & E_{43} & E_{44} \end{pmatrix}, Y_m^{2[2]_3} =$$

$$\begin{pmatrix} y_{n+1} \\ y_{n+s} \\ y_{n+r} \\ y_{n+2} \end{pmatrix}, R_1^{2[2]_3} = \begin{pmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}, R_2^{2[2]_3} = \begin{pmatrix} y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_n \end{pmatrix},$$

$$R_3^{2[2]_3} = \begin{pmatrix} y''_{n-3} \\ y''_{n-2} \\ y''_{n-1} \\ y''_n \end{pmatrix}, R_4^{2[2]_3} = \begin{pmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}, R_5^{2[2]_3} = \begin{pmatrix} f_{n+1} \\ f_{n+s} \\ f_{n+r} \\ f_{n+2} \end{pmatrix}.$$

The elements of matrix $D^{2[2]_3}$ and matrix $E^{2[2]_3}$ are given in Appendix A.

Multiplying (7) by the inverse of $A^{2[2]_3}$, we have

$$I^{2[2]_3} Y_m^{2[2]_3} = \bar{B}_1^{2[2]_3} R_1^{2[2]_3} + \bar{B}_2^{2[2]_3} R_2^{2[2]_3} + \bar{B}_3^{2[2]_3} R_3^{2[2]_3} + h^3 [\bar{D}^{2[2]_3} R_4^{2[2]_3} + \bar{E}^{2[2]_3} R_5^{2[2]_3}] \quad (8)$$

where

$$I^{2[2]_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \bar{B}_1^{2[2]_3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \bar{B}_2^{2[2]_3} = \begin{pmatrix} 0 & 0 & 0 & h \\ 0 & 0 & 0 & hs \\ 0 & 0 & 0 & hr \\ 0 & 0 & 0 & 2h \end{pmatrix},$$

$$\bar{B}_3^{2[2]_3} = \begin{pmatrix} 0 & 0 & 0 & \frac{h^2}{2} \\ 0 & 0 & 0 & \frac{h^2s^2}{2} \\ 0 & 0 & 0 & \frac{h^2r^2}{2} \\ 0 & 0 & 0 & 2h^2 \end{pmatrix}, \bar{D}^{2[2]_3} = \begin{pmatrix} 0 & 0 & 0 & \bar{D}_{14} \\ 0 & 0 & 0 & \bar{D}_{24} \\ 0 & 0 & 0 & \bar{D}_{34} \\ 0 & 0 & 0 & \bar{D}_{44} \end{pmatrix}, \bar{E}^{2[2]_3} = \begin{pmatrix} \bar{E}_{11} & \bar{E}_{12} & \bar{E}_{13} & \bar{E}_{14} \\ \bar{E}_{21} & \bar{E}_{22} & \bar{E}_{23} & \bar{E}_{24} \\ \bar{E}_{31} & \bar{E}_{32} & \bar{E}_{33} & \bar{E}_{34} \\ \bar{E}_{41} & \bar{E}_{42} & \bar{E}_{43} & \bar{E}_{44} \end{pmatrix}.$$

The elements of $\bar{D}^{2[2]_3}$ and $\bar{E}^{2[2]_3}$ are shown in Appendix B.

Equation (8) can also be written as

$$y_{n+1} = y_n + h y'_n + \frac{1}{2} h^2 y''_n + h^3 \left[\frac{1}{1680rs} (11 - 35s + 7r(-5 + 27s)) f_n + \frac{1}{1680(r^2 - 3r + 3)r(r-s)(s^2 - 3s + 3)s} 2(r-2)r(r-s)(s-2)s(10 - 21s + 7r(8s - 3)) f_{1+n} + (s-1)s(-r-1)r(r-s)(3 - 7r - 7s + 21rs) f_{2+n} - 2(s-2)(-11 + 35s) f_{n+r} + 2(r-2)(r-1)r(-11 + 35r) f_{n+s} \right], \quad (9)$$

$$y_{n+s} = y_n + hsy'_n + \frac{1}{2}h^2s^2y''_n + h^3 \left[\frac{s^3}{1680r} (7r(30 + s(s-9)) - 3s(14 + s(s-7)))f_n - \frac{1}{1680(r-2)(r-1)r(r-s)(s-2)(s-1)}(r-2)r(r-s)(-2+s)s^5(7r(-6+s) + (14-3s)s)f_{1+n} + (s-1)s^5((r-1)r(r-s)(7r(s-3) + (7-3s)s)f_{2+n} - 6(s-2)(14+(s-7)s)f_{n+r}) + 2(r-2)(r-1)rs^3(s((21-4s)s-28)+7r(10+(s-6)s))f_{n+s}, \quad (10)$$

$$y_{n+r} = y_n + hry'_n + \frac{1}{2}h^2r^2y''_n + h^3 \left[\frac{r^3}{240} ((30+r(r-9)) - \frac{3r}{7s}(14+r(r-7)))f_n - \frac{2r^3}{1680(r-2)(r-1)(s-2)(s-1)s(s-r)}(r-2)r^2(r-s)(s-2)s(3r^2+42s-7r(2+s))f_{1+n} + (r-1)r^2(r-s)(s-1)s(r(3r-7)-7(r-3)s)f_{2+n} + 2(s-2)(s-1)s(r(-28+(21-4r)r)+7(10+(r-6)r)s)f_{n+r} - 6(r-2)(r-1)r^2(14+(r-7)r)f_{n+s}, \quad (11)$$

$$y_{n+2} = y_n + 2hy'_n + 2h^2y''_n + h^3 \left[\frac{1}{105rs} (4-14s+7r(-2+9s))f_n - \frac{1}{105(r-2)(r-1)r(s-2)(s-1)s(s-r)}((4rs(r-2)(r-s)(s-2)(12-14s+7r(3s-2))f_{n+1} - s(s-1)((r-1)r(r-s)(7rs-4)f_{n+2} - 4(s-2)(7s-2)f_{n+r}) + 4(r-2)(r-1)r(7r-2)f_{n+s}). \quad (12)$$

Substituting (9) and (10) into the first and second derivatives of the discrete schemes produces first, and second derivatives of the block as follows

$$y'_{n+1} = y'_n + hy''_n + h^2 \left[\frac{1}{120rs} (3-8s+r(-8+35s))f_n + \frac{(-3+8r)}{60(r-s)s(2-3s+s^2)}f_{n+s} + \right.$$

$$\left. \frac{(4-7s+r(-7+15s))}{60(-1+r)(-1+s)}f_{n+1} + \frac{(3-8s)}{60r(2-3r+r^2)(r-s)}f_{n+r} + \frac{(-1+r(2-5s)+2s)}{120(-2+r)(-2+s)}f_{n+2} \right], \quad (13)$$

$$y'_{n+s} = y'_n + hsy''_n + h^2 \left[-\frac{s^2}{120r} (-40r+5(2+3r)s - 2(3+r)s^2+s^3)f_n + \frac{s^2(-s-2)s(2s-5)+r(20+3(s-5)s)}{60(r-s)(-2+s)(-1+s)}f_{n+s} + \frac{s^4(-2r(-5+s)+(-4+s)s)}{60(-1+r)(-1+s)}f_{n+1} - \frac{h^2s^4(10-6s+s^2)}{60r(2-3r+r^2)(r-s)}f_{n+r} - \frac{s^4(5r-2(1+r)s+s^2)}{120(-2+r)(-2+s)}f_{n+2} \right], \quad (14)$$

$$y'_{n+r} = y'_n + hry''_n + h^2 \left[-\frac{r^2(r^3-40s-2r^2(3+s)+5r(2+3s))}{120s}f_n + \frac{r^4(10-6r+r^2)}{60(r-s)s(2-3s+s^2)}f_{n+s} + \right.$$

$$\left. \frac{r^4(r^2+10s-2r(2+s))}{60(-1+r)(-1+s)}f_{n+1} + \frac{r^2(2r^3-20s-3r^2(3+s)+5r(2+3s))}{60(2-3r+r^2)(r-s)}f_{n+r} - \frac{r^4(r^2+5s-2r(1+s))}{120(-2+r)(-2+s)}f_{n+2} \right] \quad (15)$$

$$y'_{n+2} = y'_n + 2hy''_n + h^2 \left[\frac{2}{15rs} (-s+r(-1+5s))f_n + \frac{4r}{15(r-s)s(2-3s+s^2)}f_{n+s} + \right.$$

$$\left. \frac{4(4-4s+r(-4+5s))}{15(-1+r)(-1+s)}f_{n+1} + \frac{4s}{15(2r-3r^2+r^3)(-r+s)}f_{n+r} - \frac{2(-2+r+s)15(-2+r)(-2+s)f_{n+2}}{2(-2+r+s)15(-2+r)(-2+s)f_{n+2}}, \quad (16)$$

$$y''_{n+1} = y''_n + h \left[\frac{(7-15s+5r(10s-3))}{120rs}f_n + \frac{(-7+15r)}{60(r-s)s(2-3s+s^2)}f_{n+s} + \frac{(18-25s+5r(8s-5))}{60(-1+r)(-1+s)}f_{n+1} + \right.$$

$$\frac{(7-15s)}{60r(2-3r+r^2)(r-s)}f_{n+r} + \frac{(-3+r(5-10s)+5s)}{120(-2+r)(-2+s)}f_{n+2}], \quad (17)$$

$$\begin{aligned} y''_{n+s} &= y''_n \\ &+ h \left[\frac{s(5r(12+(s-6)s)+s(-20-3(-5+s)s))}{120r} f_n \right. \\ &+ \frac{s(15r(s-2)^2+s(45s-12s^2-40))}{60(r-s)(2-3s+s^2)} f_{n+s} + \\ &\left. \frac{s^3(20r-5(2+r)s+3s^2)}{60(-1+r)(-1+s)} f_{n+1} - \right. \\ &\left. \frac{s^3(20+3(-5+s)s)}{60(-2+r)(-1+r)r(r-s)} f_{n+r} + \frac{s^3(5r(-2+s)+(5-3s)s)}{120(-2+r)(-2+s)} f_{n+2} \right], \quad (18) \end{aligned}$$

$$\begin{aligned} y''_{n+r} &= y''_n \\ &+ h \left[-\frac{r(r(20+3(-5+r)r)-5(12+(-6+r)r)s)}{120s} f_n \right. \\ &+ \frac{r^3(20+3(-5+r)r)}{60(r-s)(-2+s)(-1+s)s} f_{n+s} + \\ &\left. \frac{r^3(3r^2+20s-5r(2+s))}{60(-1+r)(-1+s)} f_{n+1} + \right. \\ &\left. \frac{r(40+3r(-15+4r))-15(-2+r)^2s}{60(-2+r)(-1+r)(r-s)} f_{n+r} + \frac{r^3((5-3r)r+5(-2+r)s)}{120(-2+r)(-2+s)} f_{n+2} \right], \quad (19) \end{aligned}$$

$$\begin{aligned} y''_{n+2} &= y''_n + h \left[\frac{1}{15} \left(5 - \frac{2}{rs} \right) f_n + \right. \\ &415(r-s)s(2-3s+s^2)f_{n+s} + 415(5+ \\ &\left. \frac{1}{(-1+r)(-1+s)} \right) f_{n+1} + \\ &\left. \frac{1}{15r(2-3r+r^2)(-r+s)} f_{n+r} + \frac{1}{15} \left(5 - \frac{2}{(-2+r)(-2+s)} \right) f_{n+2} \right] \quad (20) \end{aligned}$$

2.1 Order of the Method

The linear difference operator L associated with (8) is defined as

$$\begin{aligned} L[y(x); h] &= I^{2[2]_3} Y_m^{2[2]_3} \\ &= \bar{B}_1^{2[2]_3} R_1^{2[2]_3} + \bar{B}_2^{2[2]_3} R_2^{2[2]_3} \\ &+ \bar{B}_3^{2[2]_3} R_3^{2[2]_3} \\ &+ h^3 \left[\bar{D}^{2[2]_3} R_4^{2[2]_3} \right. \\ &\left. + \bar{E}^{2[2]_3} R_5^{2[2]_3} \right] \quad (21) \end{aligned}$$

where $y(x)$ is an arbitrary test function continuously differentiable on $[a, b]$. Expanding

$Y_m^{2[2]_3}$ and $R_5^{2[2]_3}$ in Taylor's series and collecting the terms in powers of h gives

$$\begin{aligned} L[y(x); h] &= \bar{C}_0 y(x) + \bar{C}_1 h y'(x) + \bar{C}_2 h^2 y''(x) \\ &+ \dots \quad (22) \end{aligned}$$

Definition 2.1 Hybrid block method (8) and associated linear operator in (21) is said to be of order $\mathbf{d} = [d_1, d_2, d_3, d_4]^T$ if $\bar{C}_0^{2[2]_3} = \bar{C}_1^{2[2]_3} = \bar{C}_2^{2[2]_3} = \dots = \bar{C}_{d+1}^{2[2]_3} = 0$ and $\bar{C}_{d+2}^{2[2]_3} \neq 0$ with error vector constants $\bar{C}_{d+2}^{2[2]_3}$, (Refer [14]).

By using Taylor series expansion about x_n for (21), it is found that the order of method is $[5, 5, 5, 5]^T$.

The new two-step hybrid block method (8) is said to be consistent if its order greater than or equal one (1). So, the new method is consistent since its order is greater than 1.

2.2 Zero Stability

The new two-step hybrid block method in (8) and its derivatives is said to be *zero-stable* if no root of the first characteristic polynomial $\pi(\omega) = |\omega I_{4 \times 4} - \bar{B}_1^{2[2]_3}|$ is having a modulus greater than one and every root of modulus one is simple, where $I_{4 \times 4}$ is identity matrix and $\bar{B}_1^{2[2]_3}$ is the coefficients matrix of y_n function. So, if the determinant $\pi(\omega) = 0$, then

$$\begin{aligned} &= \left| \omega \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| = 0 \\ &= \omega^3(\omega - 1) = 0 \end{aligned}$$

which implies $\omega = 0, 0, 0, 1$. Hence, the newly developed method is zero stable.

2.3 Consistency and Convergence

Theorem 2.1 Consistency and zero stability are sufficient conditions for a linear multistep method to be convergent, (see [15]).

Based on the above theorem, the method in (8) is consistent and zero stable and thus is convergent.

3. Implementation

This section briefly describes the implementation of the new method to find the approximation solution of third order IVPs. First of all, the Taylor Series are employed once for developed methods to produce predicted initial values for $y_{n+i}, i = 1, 2, s, r$ where $1 < s < r < 2$. The computation is done in a block. In the first block, the obtained values are substituted in the developed methods to get the corrected values of $^l y_{n+i}, i = 1, 2, s, r$. For the second block, the value at x_{n+2} is used as the initial value and substituted in the proposed methods to yield the approximate solution. This process is repeated until it reaches the end of the integrated interval.

4. Numerical Experiments

To determine the accuracy and stability of our methods, the following third order ODEs

problems are tested for $0 < x < 10$. However, the choosing x will be depending on the existing solution and existing method to compare results in terms of error of new block methods.

The following definitions are used in the following tables.

HBM2S_{new2P}: Two-step implicit hybrid block method with generalized two off-step points based on new strategy

Olabode M1: Block method proposed by Olabode.

Olabode M2: Predictor-corrector method proposed by Olabode.

AE: Absolute error.

OSP: Off-step points used in G2SHB_{new2P}, $s = \frac{17}{16}, r = \frac{4}{3}$ for problem 1, 2, 3 and 4.

d : Order of the method.

4.1 Tested Problems

Problem1: $y''' - x + 4y' = 0, y(0) = y'(0) = 0, y''(0) = 1$ for $0 \leq x \leq 1$ with $h = \frac{1}{10}$.

Exact solution: $y(x) = \frac{3(1+\cos 2x)}{16}$.

Source: [16, 17]

Table 2: Comparison of errors obtained by new method with [16, 17] for Problem 1

x	AE in HBM2S _{new2P}	AE in Adesanya(2011)	AE in Ayowemi et. al., (2014)
	OSP	$d = 7$	$d = 7$
0.1	1.292391e-10	1.189944e-11	1.1899e-11
0.2	7.581347e-10	3.042207e-09	3.0422e-09
0.3	1.953290e-09	7.779556e-08	7.7796e-08
0.4	3.942136e-09	7.746692e-07	1.5559e-07
0.5	6.705333e-09	4.599010e-06	3.0541e-07
0.6	1.030812e-08	6.478349e-06	4.6102e-07
0.7	1.462561e-08	5.783963e-06	3.1380e-07
0.8	1.953372e-08	2.354715e-06	7.0374e-07
0.9	2.481137e-08	3.766592e-06	1.0177e-06
1.0	3.016123e-08	1.233120e-05	1.6528e-06

Problem 2: $y''' - e^x = 0$, $y(0) = 3, y'(0) = 1, y''(0) = 5$ for $0 \leq x \leq 1$ with $h = \frac{1}{10}$.

Exact solution: $y(x) = 2x^2 + e^x + 2$.

Source: [18, 19]

Table 2: Comparison of errors obtained by new method with [18, 19] for Problem 2

x	AE in HBM2S _{new2P}	AE in Kuboye and Omar (2015)	AE in Olabode and Yusuph (2009)
	OSP	$d = 6$	$d = 5$
0.1	3.023359e-12	3.369305e-12	7.56479e-11
0.2	1.789324e-11	2.160050e-11	1.83983e-9
0.3	4.703704e-11	5.333245e-11	4.42400e-9
0.4	1.002545e-10	9.988632e-11	1.03587e-8
0.5	1.805089e-10	1.598988e-10	1.12999e-8
0.6	2.997727e-10	2.511404e-10	1.46095e-8
0.7	4.616627e-10	3.961489e-10	2.05295e-8
0.8	6.808030e-10	5.926823e-10	1.95075e-8
0.9	9.616121e-10	8.429168e-10	1.08431e-8
1.0	1.321949e-09	1.144603e-09	1.54095e-7

Problem 3: $y''' - 3 \sin x = 0$, $y(0) = 1, y'(0) = 0, y''(0) = -2$ for $0 \leq x \leq 1$ with $h = \frac{1}{10}$.

Exact solution: $y(x) = \frac{x^2}{2} + 3 \cos x$.

Source: [20]

Table 3: Comparison of errors obtained by new method with [20] for Problem 3

x	AE in HBM2S _{new2P}	AE in OlabodeM1 (2013)	AE in OlabodeM2 (2013)
	OSP	$d = 8$	$d = 8$
0.1	8.213319e-12	1.65922e-10	4.172279744e09
0.2	4.857537e-11	4.76275e-10	9.578546178e-08
0.3	1.256936e-10	6.23182e-10	3.991586710e-07
0.4	2.599922e-10	19.9134e-10	1.036864440e-06
0.5	4.556018e-10	3.28882e-10	2.128509889e-06
0.6	7.312086e-10	1.27096e-09	3.789539851e-06
0.7	1.090302e-09	4.84653e-09	6.130086676e-06
0.8	1.549085e-09	1.09585e-08	9.253867047e-06
0.9	2.110265e-09	2.01880e-08	1.325714643e-05
1.0	2.786920e-09	3.53956e-08	1.822777782e-05

Problem 4: $y''' - y'(2xy'' + y') = 0$, $y(0) = 1, y'(0) = \frac{1}{2}, y''(0) = 0$ for $0 \leq x \leq 1$

with $h = \frac{1}{100}$.

Exact solution: $y(x) = 1 + \frac{1}{2} \ln \left| \frac{2+x}{2-x} \right|$.

Source: [21, 16]

Table 4: Comparison of errors obtained by new method with [21, 16] for Problem 4

x	AE in HBM2S _{new2P}	AE in Ogunware et al., (2018)	AE in Adesanya (2011)
	OSP	$d = 6$	$d = 5$
0.21	1.110223e-15	7.178702e-13	8.037948 e-11
0.31	4.218847e-15	2.444489e-12	6.043090 e-10
0.41	1.110223e-14	6.052270e-12	2.581908e-09
0.51	2.309264e-14	1.270273e-11	8.158301e-09
0.61	4.130030e-14	2.417755e-11	2.141286e-08
0.71	6.861178e-14	4.339396e-11	4.969641e-08
0.81	1.079137e-13	7.532197e-11	1.620387e-07

5. Discussion and Conclusions

A new two-step hybrid block method with new strategy to solve the third order ordinary differential equations directly has been successfully developed. The new method possesses good properties of numerical method and has an order of five. The performance of the new method has proven to be compatible and better than the existing methods when solving the same problems.

6. Acknowledgments

The authors gratefully acknowledge the financial support provided by Ministry of Higher, Malaysia under FRGS grant (SO Code: 13818).

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Appendix A

$$D_{14} = - \frac{(r-1)(3r^5 + r^2(24 - 154s) + 8r^3(3 + 7s) - r^4(18 + 7s) + s(11 - 24s - 24s^2 + 18s^3 - 3s^4) + r(154s^2 - 11 - 56s^3 + 7s^4))}{1680s}$$

$$D_{24} = \frac{(s-2)(24s - 12s^3 - 5 + 3s^4 - 7r(-3 + 10s - 6s^2 + s^3))}{840rs}, D_{34} = \frac{(11 - 24s - 24s^2 + 18s^3 - 3s^4 + 7r(22s - 8s^2 - 5 + s^3))}{1680hr}$$

$$D_{44} = - \frac{(11 - 24s - 24s^2 - 24s^3 + 18s^4 - 3s^5 + 7r(-5 + 22s + 22s^2 - 8s^3 + s^4))}{840h^2rs}$$

$$E_{11} = \frac{r(3r^5 - r^4(11 + 7s) + r^2(-11 + 35s) + r^3(-11 + 35s) + s(-10 + 11s + 11s^2 + 11s^3 - 3s^4) + r(10 - 35s^2 - 35s^3 + 7s^4))}{840(-1 + s)}$$

$$E_{12} = \frac{(r-1)r(-11 + 3r^4 + 3r^3(-6 + s) - 11s - 11s^2 + 17s^3 - 4s^4 + 3r^2(8 - 6s + s^2) + 3r(8 + 8s - 6s^2 + s^3))}{840s(2 - 3s + s^2)}$$

$$E_{13} = \frac{(11 + 4r^4 - 24s - 24s^2 + 18s^3 - 3s^4 - r^3(17 + 3s) + r^2(11 + 18s - 3s^2) + r(11 - 24s + 18s^2 - 3s^3))}{840(-2 + r)},$$

$$E_{14} = -\frac{(r - 1)r(3r^5 + 2r^2(7s - 2) + 2r^3(7s - 2) - r^4(4 + 7s) + s(-3 + 4s + 4s^2 + 4s^3 - 3s^4) + r(3 - 14s^2 - 14s^3 + 7s^4))}{1680(-2 + r)(-2 + s)},$$

$$E_{21} = \frac{(s - 2)(53s + 21s^2 + 5s^3 - 86 - 3s^4 + 7r(13 - 11s - 3s^2 + s^3))}{420(-1 + r)(-1 + s)}, E_{22} = \frac{(3s + 7s^2 + 9s^3 - 5 - 4s^4 + 7r(3 - s - 3s^2 + s^3))}{420(-1 + s)s(-r + s)},$$

$$E_{23} = \frac{(10 - 53s + 24s^2 + 24s^3 - 18s^4 + 3s^5)}{420r(2 - 3r + r^2)(r - s)}, E_{24} = \frac{(-19 - 4s + 2s^3 + 3s^4 - 7r(-1 - 2s + s^3))}{840(-2 + r)},$$

$$E_{31} = \frac{s(10 - 11s - 11s^2 - 11s^3 + 3s^4 - 7r(3 - 5s - 5s^2 + s^3))}{840h(-1 + r)(-1 + s)}, E_{32} = \frac{(11 + 11s + 11s^2 - 17s^3 + 4s^4 - 7r(5 + 5s - 5s^2 + s^3))}{840h(-2 + s)(-1 + s)(-r + s)},$$

$$E_{33} = -\frac{s(-11 + 24s + 24s^2 - 18s^3 + 3s^4)}{840hr(2 - 3r + r^2)(r - s)}, E_{34} = \frac{s(-3 + 4s + 4s^2 + 4s^3 - 3s^4 + 7r(1 - 2s - 2s^2 + s^3))}{1680h(-2 + r)(-2 + s)},$$

$$E_{41} = \frac{(-10 + 11s + 11s^2 + 11s^3 + 11s^4 - 3s^5 + 7r(3 - 5s - 5s^2 - 5s^3 + s^4))}{420h^2(-1 + r)(-1 + s)},$$

$$E_{42} = \frac{(11 + 11s + 11s^2 + 11s^3 - 17s^4 + 4s^5 - 7r(5 + 5s + 5s^2 - 5s^3 + s^4))}{420h^2(r - s)s(2 - 3s + s^2)}, E_{43} = \frac{(-11 + 24s + 24s^2 + 24s^3 - 18s^4 + 3s^5)}{420h^2r(2 - 3r + r^2)(r - s)},$$

$$E_{44} = -\frac{(-3 + 4s + 4s^2 + 4s^3 + 4s^4 - 3s^5 + 7r(1 - 2s - 2s^2 - 2s^3 + s^4))}{840h^2(-2 + r)(-2 + s)}.$$

Appendix B

$$\bar{D}_{14} = \frac{s^4(s(-56-3(-8+s)s)+4r(84+s(-21+2s)))}{10080r}, \bar{D}_{24} = \frac{(13-48s+4r(-12+77s))}{10080rs},$$

$$\bar{D}_{34} = \frac{r^4(-3r^3+336s+8r^2(3+s)-28r(2+3s))}{10080s}, \bar{D}_{44} = \frac{8(1-3s+r(-3+14s))}{315rs},$$

$$\bar{E}_{11} = \frac{s^4(s(-28+18s-3s^2)+6r(14-7s+s^2))}{5040(r-s)(2-3s+s^2)}, \bar{E}_{12} = \frac{s^6(-8r(-7+s)+s(-16+3s))}{5040(-1+r)(-1+s)},$$

$$\bar{E}_{13} = -\frac{s^6(56-24s+3s^2)}{5040r(2-3r+r^2)(r-s)}, \bar{E}_{14} = \frac{s^6((8-3s)s+4r(-7+2s))}{10080(-2+r)(-2+s)}, \bar{E}_{21} = \frac{-13+48r}{5040(r-s)s(2-3s+s^2)},$$

$$\bar{E}_{22} = \frac{9-22s+r(-22+70s)}{5040(-1+r)(-1+s)}, \bar{E}_{23} = \frac{13-48s}{5040r(2-3r+r^2)(r-s)}, \bar{E}_{24} = \frac{-3+r(8-28s)+8s}{10080(-2+r)(-2+s)},$$

$$\bar{E}_{31} = \frac{r^6(56-24r+3r^2)}{5040(r-s)s(2-3s+s^2)}, \bar{E}_{32} = \frac{r^6(3r^2+56s-8r(2+s))}{5040(-1+r)(-1+s)}, \bar{E}_{33} = \frac{r^4(3r^3-84s-6r^2(3+s)+14r(2+3s))}{5040(2-3r+r^2)(r-s)},$$

$$\bar{E}_{34} = \frac{r^6(-3r^2-28s+8r(1+s))}{10080(-2+r)(-2+s)}, \bar{E}_{41} = \frac{16(-1+3r)}{315(r-s)(-2+s)(-1+s)s}, \bar{E}_{42} = \frac{16(3-4s+r(-4+7s))}{315(-1+r)(-1+s)},$$

$$\bar{E}_{43} = -\frac{16(-1+3s)}{315r(2-3r+r^2)(r-s)}, \bar{E}_{44} = \frac{r(4-14s)+4s}{315(-2+r)(-2+s)}.$$