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Reliability Estimation after Selection from one Parameter Exponential Population

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Abstract:
Let Π1,Π2,...,Πk be k populations, where Πi being exponential with unknown hazard rate λi, i =1,...,k. Suppose independent random samples are drawn from populations Π1,Π2,...,Πk. Let Xi1,Xi2,...,Xin, i =1,...,k. be a random sample of size n drawn from the ithpopulation. Let Xi = ∑_(j=1)^n▒X_ij be the sample mean of ithpopulation. The natural selection rule is to select the population with the highest mean. That is, Πi is selected, if Xi = max(X1, . . . ,Xk). We consider the problem of estimating the Reliability function of the selected population. The Unique Minimum Variance Unbiased Estimator(UMVUE) is derived and some natural estimators are proposed. Finally a numerical comparison of the risks of these estimators is done when the loss function is squared error.

1. Introduction

The problem of estimation of reliability function in exponential population has been extensively studied in the literature. To see a detailed review in this area, one may see Kumar et al.[1], Mahapatra et al.[2]. However, the estimation of Reliability function(Survival function) of a selected exponential population has not been addressed so much in the literature. Kumar et al.[1] have studied the estimation of the reliability function after a subset selection for a two parameter exponential population where the failure rate is known. They have derived the uniformly minimum variance unbiased estimator and some natural estimators. These estimators are further improved by solving a differential inequality. To the best of our knowledge probably, that was the only work available in the literature. This present work considers the estimation of the survival function after selection from exponential population. It may be noted that the model used by Kumar et al.[1] is different to this model. However, the estimation of a parameter from a selected population is quite useful in "Ranking and Selection Methodology". One may be interested to purchase a car having high reliability or low hazard rate. Then, he may also be interested to know the reliability or hazard rate of the car that he has purchased for his personal use. For some useful references in this context, one may see Sackrowitz and Samuel-Cahn [3], Vellaisamy and Sharma[4], [5], Arshad and Misra[6], Arshad et al.[7]. The estimation of a scale parameter in exponential distribution was probably first studied by Sackrowitz and Samuel-Cahn[3] for k=2 exponential populations. Independent samples of size 1 were taken from each population. Two selection rules were considered based upon maximum or minimum observation. Later, Vellaisamy and Sharma[4] have considered the estimation of the reverse hazard rate $\frac{1}{\lambda_i}$ after selection from a general Gamma population. It was assumed that the shape parameters are known positive integers. They derived the UMVUE of the

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selected scale parameter. An admissible class of estimators was constructed using Brewster-Zidek [8] technique for k = 2 populations. Misra et al.[9] have extended this work for the case when the shape parameters are known positive real numbers. They have also studied for the case k =2 populations. This paper is organized as follows: Some preliminary notations and the selection rules are presented in Section 1. The UMVUE of the selected population is derived and some natural estimators are proposed in Section 2 and 2.1. The risks of these estimators are compared through a simulation study in Section 3.

2. Preliminaries

In this section, we derive the estimand from a exponential population when the inverse of the scale parameters are unknown. Let $\Pi_1, \ldots, \Pi_k$ be k populations, where Π_i has the density

$$f_i(x) = \lambda_i e^{-\lambda_i x},\ x > 0,\ \lambda_i > 0,\ i = 1, \ldots, k.$$

Then the survival function $\theta_i(t)$ at time t >0, of the i^{th} population is derived as

$$\theta_i(t) = P(X_{ij} > t) = e^{-\lambda_i t}. \qquad (2.1)$$

Let independent random samples are drawn from each of the populations $\Pi_1,\ldots,\Pi_k$. Let $X_{i1},\ldots,X_{in}$ be a random sample of size n drawn from i^{th} population, $i = 1,\ldots,k$. Let $X_i = \sum_{j=1}^{n} X_{ij}$ then we observe that $\underline{X} = (X_1, \ldots, X_k)$ is complete and sufficient for $\underline{\lambda} = (\lambda_1, \ldots, \lambda_k)$. Suppose $X_{(1)} \geq \ldots \geq X_{(k)}$ be the order statistics of $X_1, \ldots, X_k$. Note that X_i has Gamma (n, λ_i) distribution with density

$$\frac{\lambda_i^n}{\Gamma(n)} x^{n-1} e^{-\lambda_i x}, \qquad x > 0, \qquad \lambda_i > 0.$$

We want to select the population having largest X_i. The optimality of this decision rule was studied by Gupta and Panchapakesan[10]. We want to estimate $\theta_i(t)$ after selection from these k populations according to a selection rule. More precisely, we want to estimate $\theta_M = \sum_{i=1}^{k} \theta_i I_i$, where $I_i = 1, \text{if } X_i > X_{(1)i}$,

$= 0, \text{otherwise}, \quad i = 1,2,3, \ldots \ldots k.$

and

$$X_{(1)i} = \max\{X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_k\}.$$

The case of ties is ignored, since the distribution under consideration is continuous.

2.1. Derivation of the UMVUE and Some Natural Estimators

In order to derive the UMVUE, we prove the followings:

Lemma 2.1. Let $X \sim \text{Gamma}(n,\lambda)$. Define $I_B(x)$ as the usual indicator function. Then for any a >0,

$$E[e^{-\lambda t} I_{(a,\infty)}(X)] = E[(\tfrac{X-t}{X})^{n-1} I_{(a+t,\infty)}(X)].$$

Proof: Note for any $\alpha > 0$, we have

$E[e^{-\lambda t} I_{(a,\infty)}(X)] = \int_a^\infty x^{n-1} e^{-\lambda(x+t)}\, dx$

$= \int_{a+t}^\infty (x-t)^{n-1} e^{-\lambda x}\, dx$

$= \int_{a+t}^\infty \left(\frac{(x-t)}{x}\right)^{n-1} x^{n-1} e^{-\lambda x}\, dx$

which shows $E[e^{-\lambda t} I_{(a,\infty)}(X)] = E[(\tfrac{X-t}{X})^{n-1} I_{(a+t,\infty)}(X)]$. This proves the lemma.

We can extend Lemma 3.1 to the case of k independent variables $X_1, X_2, X_3, \ldots, X_k$, where X_i follows $\text{Gamma}(n, \lambda_i)$, $i = 1,2,3,\ldots, k$. Then, we prove the following result.

Lemma 2.2.

$$E\left[\sum_{i=1}^{k} e^{-\lambda_i t} I_i\right] = E\left[\sum_{i=1}^{k} \left(1 - \frac{t}{X_i}\right)^{n-1} I_{(X_{(1)i}+t,\infty)}(X_i)\right]$$

Proof: Applying Lemma 3.1, we can write

$E[e^{-\lambda_i t} I_i] = E\left(1 - \frac{t}{X_i}\right)^{n-1} I_{(X_{(1)i}+t,\infty)}(X_i),$

which implies

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$$E\left[\sum_{i=1}^{k} e^{-\lambda_i t}\ I_i\right] = E\left[\sum_{i=1}^{k}\left(1-\frac{t}{X_i}\right)^{n-1} I_{(X_{(1)i}+t,\infty)}(X_i)\right]$$

This proves the lemma. Lemma 3.2 immediately yields the following result.

Theorem 2.1. The UMVUE of θM is given by

$$\delta_U = \left(1-\frac{t}{X_{(1)}}\right)^{n-1} I_{(X_{(2)}+t,\infty)}(X_{(1)})$$

It may be seen that the UMVUE of the hazard rate λi does not exist for this problem (see Vellaisamy and Jain [11]) but it is interesting to see that the UMVUE exists for the reliability function. The form of this estimator is quite different from the UMVUE of reliability function for the component problem(see Zacks and Even[12]). Next we obtain some natural estimators of θM . It can be easily seen that the Maximum Likelihood Estimators(MLE) of $\theta i(t)$ for the component problem is $e^{-\frac{nt}{X_i}}$ (that is based upon the i^{th} sample only). So, we propose a natural estimator, which is the MLE of θM , given by$\delta_{ML} = e^{-\frac{nt}{X_{(1)}}}$. Further, it can be seen that the "best scale equivariant estimator" for the component problem is $e^{-\frac{(n-2)t}{X_i}}$. So an analogue of the best scale equivariant estimator is given by $\delta_N = e^{-\frac{(n-2)t}{X_{(1)}}}$.We consider the squared error loss function. It is given by $L(\theta,\delta)= (\theta - \delta)^2$, where θ is unknown and δ is an estimator of θ. The form of these two estimators are such that the exact risk (Mean Squared Error) is difficult to derive. So the exact decision theoretic properties are also hard to study. In the next section, the expected losses of these estimators are compared through a numerical approach.

3. Numerical Comparison In this section, we have numerically compared the risks of the estimators δN , δU and δMLE for some choices of $\lambda 1$, $\lambda 2$ with respect to loss as defined in the previous section. For the computationalpurpose, t = 1 and k = 2 has been taken. The risks of these estimators are tabulated by taking different values of n as shown in the following tables. The risk values are based on 10,000 samples of sizes n from the exponential populations for given values of $\lambda_1$and λ_2. The following points are observed.

(i) The mean squared errors of these estimators decrease when n increases. This is true for all values of $\lambda_1$and λ_2.

(ii) For large sample size, the risks of the estimators $\delta_N$and $\delta_{MLE}$are same. Since the estimator $\delta_U$is not smooth, the performance of this estimator does not follow the same pattern. However the UMVUE dominates the other two estimators in some regions of the parameter space.

(iii) The estimator $\delta_N$dominates the other two estimators in terms of risk for large n and for some large values of $\lambda_1$and λ_2 .

(iv) Moreover, it seems that the estimator δ_{MLE}has a better performance over the other two estimators over a substantial portion of the parameter space.

(v) From the simulation study, we are unable to establish a hierarchy among these estimators. However, we recommend the MLE as it dominates the other two estimators over a substantial portion of the parameter space. Certainly the UMVUE is not preferred unless one is interested in the class of unbiased estimators.

(vi) We have calculated the risks of these estimators for various values of n, $\lambda_1$and λ_2. However, all the risk values are not incorporated in the table.

Table 1: n =5

↓	↓	$R(\delta_N)$	$R(\delta_U)$	(δ_{ML})
	25	0.010598	0.050013	005053
	5	0.013310	0.033608	008781
	75	0.010692	0.021662	011133
25	0	0.010777	0.014945	012449
	5	0.009390	0.011467	013150
	0	0.009467	0.010186	013133
	25	0.013437	0.033545	009053
	5	0.029786	0.077278	012082
	75	0.034197	0.055912	015328
5	0	0.032327	0.047723	017653
	5	0.025477	0.037423	021157
	0	0.023998	0.026255	022440
	25	0.010621	0.017370	012746
	5	0.032669	0.043255	018061
	75	0.050816	0.058729	018647
0	0	0.059791	0.062297	018849
	5	0.059770	0.046842	019951
	0	0.052402	0.040990	020393
	25	0.009385	0.012493	013187
	5	0.028200	0.034428	021704
	75	0.046188	0.046198	021367
5	0	0.060271	0.047723	020282
	5	0.068961	0.041403	017941
	0	0.064721	0.034147	016712

Table 2: n =15

↓	↓	$R(\delta_N)$	$R(\delta_U)$	(δ_{ML})
	25	0.002053	0.035868	001480
	5	0.002480	0.002972	003388
	75	0.002245	0.002756	003401
25	0	0.002232	0.002746	003381
	5	0.002225	0.002743	003369
	0	0.002217	0.002729	003338
	25	0.002603	0.002918	003248
	5	0.005344	0.045147	003576
	75	0.006623	0.028832	005612
5	0	0.005789	0.006654	007305
	5	0.004897	0.006548	008217
	0	0.005789	0.006657	008205
	25	0.002251	0.002762	003406
	5	0.005953	0.008660	006974
	75	0.006860	0.022507	005201
0	0	0.009120	0.038221	005320
	5	0.009515	0.017069	007082
	0	0.008294	0.011074	008830
	25	0.002251	0.002763	003406
	5	0.005402	0.006512	007408

	75	0.007837	0.008978	008548
5	0	0.008072	0.019080	007019
	5	0.008838	0.019210	004560
	0	0.008247	0.010156	004942

Table 3: n = 20

↓	↓	$R(\delta_N)$	$R(\delta_U)$	$R(\delta_{ML})$
	25	0.001838	0.021601	0.001213
	5	0.001918	0.003208	0.002155
	75	0.001789	0.001942	0.002192
25	0	0.001743	0.001654	0.002096
	5	0.001682	0.002970	0.001961
	0	0.001549	0.002674	0.001794
	25	0.001682	0.002970	0.001961
	5	0.004789	0.030114	0.003038
	75	0.004335	0.020102	0.004027
5	0	0.004546	0.005414	0.004807
	5	0.004340	0.004611	0.004922
	0	0.004295	0.004372	0.004984
	25	0.001786	0.001943	0.002197
	5	0.004095	0.005857	0.004461
	75	0.006698	0.014778	0.004853
0	0	0.008190	0.024909	0.004841
	5	0.006644	0.013585	0.005505
	0	0.006698	0.007435	0.006224
	25	0.001786	0.001943	0.002197
	5	0.004328	0.004611	0.004929
	75	0.005673	0.006557	0.005765
5	0	0.007157	0.011118	0.005451
	5	0.007957	0.014818	0.004421
	0	0.006700	0.010669	0.004299

4. Conclusion

The present problem investigates the estimation of survival function of a selected exponential population. Interestingly, we are able to get the UMVUE which has not been discussed in the literature so far.

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