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Variant Second order method for the numerical solution Initial Value Problem in Ordinary Differential Equations

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Abstract:
In this paper an attempt has been made to find an alternative numerical method for the solution of the initial value problem in ordinary differential equations. Previously, researchers have used Taylor's series expansion or they have approximated the definite integral by different quadrature rules to develop numerical methods for the said purpose. We have developed a variant method using numerical integration based on Haar wavelet to approximate the same. The performance and the stability of the constructed method have been studied. The constructed method is found to be of second order.

***Keywords**: Quadrature rule, Initial value problem, Euler method, Runge-Kutta method, stability.*

INTRODUCTION

We come across many physical problems in the fields of science and engineering. The solutions of those problems are very much desired. For that purpose these problems are mathematically formulated, called mathematical models. The models can be of different types, namely system of linear or nonlinear equations, ordinary differential equations, partial differential equations and many more. So we need to have tools to solve the models. Unfortunately, many of the ordinary differential equations do not have analytic solution and many of them are too difficult to be solved analytically. Hence we must have efficient numerical methods to handle them. In late 19th century and 20th century many a methods and algorithms were developed to solve the initial value problem in ordinary differential equations numerically. Additionally, the advent of electronic computers in the mid 20th has made the job easier for those numerical methods and the use of these methods became extensive.

Let the initial value problem be

$$y' = f(x,y), y(x_o) = y_0 (1)$$

There are a good number of efficient methods to find the numerical solution of above initial value problem (1). Euler' s method[4] is fundamental in this effort.The stability and the accuracy of Euler's method is very low[6]. Runge[2], Kutta[3], Heun[1] and Nyström [12] are some early works with improved stability and accuracy. Later Butcher[7], Gill[8], Butcher and Warner[5] and Merson[13] have contributed to the advancement of those ideas.We are presenting a few of them.

1) 1.1 Euler's method

$$y_{n+1} = y_n + hf(x_n, y_n) \quad (2)$$

The scheme is developed by using the Newton's forward difference interpolation. This is a first order method.

2) 1.2 Taylor's method

$$y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2}(f_x + f_y f)(x_n, y_n) + \cdots \quad (3)$$

3) 1.3 Modified Euler's method

$$y_{n+1} = y_n + hk_2 \quad (4)$$

where $k_1 = f(x_n, y_n)$ and $k_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1)$.

The scheme has been developed by using mid-point rule. This is a 2nd order method.

4) 1.4 Improved Euler's method

$$y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2) \quad (5)$$

where $k_1 = f(x_n, y_n)$ and $k_2 = f(x_n + h, y_n + hk_1)$

The scheme has been developed by using trapezoidal rule. This is a 2nd order method.

5) 1.5 Heun's method

$$y_{n+1} = y_n + \frac{(k_1 + k_2)}{2} \quad (6)$$

where $k_1 = hf(x_n, y_n)$ and $k_2 = hf(x_n + h, y_n + k_1)$

The scheme has been developed by using trapezoidal rule. This is a 2nd order method.

6) 1.6 Runge-Kutta 2nd order method

$$y_{n+1} = y_n + \frac{(2k_1 + k_2)}{3} \quad (7)$$

where $k_1 = hf(x_n, y_n)$ and $k_2 = hf(x_n + \frac{3}{2}h, y_n + \frac{3}{2}k_1)$

The scheme has been developed by comparing with Taylor's series to achieve 2nd order accuracy. This is a 2nd order method.

7) 1.7 Runge-Kutta 4th order method

$$y_{n+1} = y_n + \frac{(k_1 + 2k_2 + 2k_3 + k_4)}{6} \quad (8)$$

where $k_1 = hf(x_n, y_n)$,$k_2 = hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1)$,$k_3 = hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2)$
and $k_4 = hf(x_n + h, y_n + k_3)$.

The scheme has been developed by comparing with Taylor's series to achieve 4th order accuracy. This is a 4th order method.

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Consequently, some modifications are also made to the above methods to improve the accuracy of the solution. Recently, Abraham O [10], [11] has improved the modified Euler's method. But the classical Runge-Kutta 4th order method do stand handy to use.

Now, we have developed a variant one step method to solve the IVP numerically using numerical integration based on Haar-wavelets [9]. The constructed method is found to be of 2nd order. Further, we have found the stability region for this method and evaluated the error associated with the method .

The paper is organized as follows. In Section-2 a new modern one step variant method is developed .Section-3 presents the order of the new variant method. In Section-4 Stability of the method is studied. Numerical examples are provided in Section-5. The last section, Section-6 concludes the paper.

2 Main results

New method

Let us solve the initial value problem (1) as follows;

$$\frac{dy}{dx} = f(x, y)$$

i.e.

$$dy = f(x, y)dx$$

Now integrating both sides from x_0 to $x_0 + \alpha h$ we have

$$\int_{x_0}^{x_0+\alpha h} dy = \int_{x_0}^{x_0+\alpha h} f(x, y)dx$$

where $0 < \alpha \leq 1$

i.e.

$$y(x_0 + \alpha h) = y(x_0) + f(x_0, y_0) \int_{x_0}^{x_0+\alpha h} dx$$

(Choosing $f(x, y) = f(x_0, y_0)$, for $x_0 \leq x \leq x_0 + \alpha h$)

So

$$y(x_0 + \alpha h) = y(x_0) + \alpha h f(x_0, y_0)$$

In general, we have

$$y(x_n + \alpha h) = y(x_n) + \alpha h f(x_n, y_n) \tag{9}$$

Again let us solve

$$\frac{dy}{dx} = f(x, y), y(x_n) = y_n \tag{10}$$

as follows.

$$\frac{dy}{dx} = f(x, y)$$

i.e

$$\int_{x_n}^{x_n+\alpha h} dy = \int_{x_n}^{x_n+\alpha h} f(x, y)dx$$

Using the quadrature rule by Islam et al.[9]

$$y(x_n+\alpha h)=y(x_n)+\frac{(x_n+\alpha h)-(x_n)}{2}\sum_{k=1}^{2} f(x_n+\frac{(\alpha h)(k-0.5)}{2},y\left(x_n+\frac{(\alpha h)(k-0.5)}{2}\right)$$
$$=y_n+\frac{\alpha h}{2}(f(x_n+\frac{\alpha h}{4},y(x_n+\frac{\alpha h}{4}))+f(x_n+\frac{3\alpha h}{4},y(x_n+\frac{3\alpha h}{4})))$$
$$=y_n+\tfrac{\alpha h}{2}(f(x_n+\tfrac{\alpha h}{4},y(x_n)+\tfrac{\alpha h}{4})+f(x_n+\tfrac{3\alpha h}{4},y(x_n)+\tfrac{3\alpha h}{4})) \quad (11)$$

where $f_n=f(x_n,y_n)$ and using (9)

For $\alpha=1$,

$$y(x_n+h)=y(x_{n+1})$$
$$=y_{n+1}$$
$$=y_n+\tfrac{h}{2}(f(x_n+\tfrac{h}{4},y(x_n)+\tfrac{h}{4}f_n)+f(x_n+\tfrac{3h}{4},y(x_n)+\tfrac{3h}{4}f_n)) \quad (12)$$

Above method is named as Variant Method (VM) for the solution of IVP in ordinary differential equations.

3 Order of the Variant method

Using Taylor's formula:

$$y(x_n+h)-y(x_n)=hy'(x_n)+\frac{h^2}{2!}y''(x_n)+\frac{h^3}{3!}y'''(x_n)+\cdots$$
$$=hf(x_n,y_n)+\tfrac{h^2}{2}(f_x+f_nf_y)+\tfrac{h^3}{6}(f_{xx}+2f_nf_{xy}+f_n{}^2f_{yy}+f_xf_y+f_nf_y{}^2)+\cdots \quad (13)$$

And using the Variant method formula (12), we have

$$y_{n+1}-y_n=\tfrac{h}{2}[(f(x_n,y_n)+\tfrac{h}{4}f_x(x_n,y_n)+\tfrac{h}{4}f_nf_y(x_n,y_n)+\tfrac{h^2}{32}f_{xx}(x_n,y_n)$$
$$+\tfrac{h^2}{16}f_nf_{xy}(x_n,y_n)+\tfrac{h^2}{32}f_n{}^2f_{yy}(x_n,y_n)+O(h^3))$$
$$+(f(x_n,y_n)+\tfrac{3h}{4}f_x(x_n,y_n)+\tfrac{3h}{4}f_nf_y(x_n,y_n)+\tfrac{9h^2}{32}f_{xx}(x_n,y_n)$$
$$+\tfrac{9h^2}{16}f_nf_{xy}(x_n,y_n)+\tfrac{9h^2}{32}f_n{}^2f_{yy}(x_n,y_n)+O(h^3))]$$
$$=hf(x_n,y_n)+\tfrac{h^2}{2}(f_x(x_n,y_n)+f_nf_y(x_n,y_n))$$
$$+\tfrac{5h^3}{32}(f_{xx}(x_n,y_n)+2f_nf_{xy}(x_n,y_n)+f_n{}^2f_{yy}(x_n,y_n))+O(h^4)$$
$$=hf(x_n,y_n)+\tfrac{h^2}{2}f'(x_n,y_n)$$
$$+\tfrac{5h^3}{32}(f_{xx}(x_n,y_n)+2f_nf_{xy}(x_n,y_n)+f_n{}^2f_{yy}(x_n,y_n))+O(h^4) \quad (14)$$

From the equations (13) and (14), we see that

$$ERROR=\tfrac{h^3}{6}(f_{xx}+2f_nf_{xy}+f_n{}^2f_{yy}+f_xf_y+f_nf_y{}^2)$$
$$-\tfrac{5h^3}{32}(f_{xx}+2f_nf_{xy}+f_n{}^2f_{yy})+O(h^4)$$
$$=\tfrac{h^3}{6}(f_{xx}+2f_nf_{xy}+f_n{}^2f_{yy}+f_y(f_x+f_nf_y))+$$
$$-\tfrac{5h^3}{32}(f_{xx}+2f_nf_{xy}+f_n{}^2f_{yy})+O(h^4)$$
$$=h^3(\tfrac{1}{96}f_{xx}+\tfrac{1}{48}f_nf_{xy}+\tfrac{1}{96}f_n{}^2f_{yy}+\tfrac{1}{6}f_y(f_x+f_nf_y))+O(h^4)$$
$$=O(h^3) \quad (15)$$

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It is seen that the variant method agrees upto $O(h^2)$. Hence we affirm that the constructed method is of second order and the local error is $O(h^3)$.

4 Stability of the Variant method

In this section the numerical stability of the proposed method is discussed. And it is compared with the stability of the contemporary methods.

It is quite possible that the numerical solution of a differential equation may grow unbounded even though its exact solution is well-behaved. To analyze this let us consider the initial value problem :

$$\frac{dy}{dx} = f(x,y), y(x_n) = y_n$$

Our objective is to get a numerically stable solution to the given initial value problem and this can be achieved by determining the range of the step size such that the numerical solution remains bounded.

Now we consider the Taylor's series for a function of two-variables:

$$f(x,y) = f(x_n,y_n) + (x - x_n)\frac{\partial f}{\partial x}(x_n,y_n) + (y - y_n)\frac{\partial f}{\partial y}(x_n,y_n)$$
$$+\frac{1}{2!}[(x - x_n)^2\frac{\partial^2 f}{\partial x^2}(x_n,y_n)$$
$$+2(x - x_n)(y - y_n)\frac{\partial^2 f}{\partial x\partial y}(x_n,y_n)$$
$$+(y - y_n)^2\frac{\partial^2 f}{\partial y^2}(x_n,y_n)] + \cdots \quad (16)$$

Considering only the linear terms of the above series and substituting the same in the given IVP, we get

$$\frac{dy}{dx} = f(x_n,y_n) + (x - x_n)\frac{\partial f}{\partial x}(x_n,y_n) + (y - y_n)\frac{\partial f}{\partial y}(x_n,y_n)$$
$$= f(x_n,y_n) + x\frac{\partial f}{\partial x}(x_n,y_n) + y\frac{\partial f}{\partial y}(x_n,y_n)$$
$$-[x_n\frac{\partial f}{\partial x}(x_n,y_n) + y_n\frac{\partial f}{\partial y}(x_n,y_n)]$$
$$= y\frac{\partial f}{\partial y}(x_n,y_n) + x\frac{\partial f}{\partial x}(x_n,y_n)$$
$$+f(x_n,y_n) - [x_n\frac{\partial f}{\partial x}(x_n,y_n) + y_n\frac{\partial f}{\partial y}(x_n,y_n)]$$
$$= \alpha y + \beta x + \gamma \quad (17)$$

where

$$\alpha = \frac{\partial f}{\partial y}(x_n,y_n), \beta = \frac{\partial f}{\partial x}(x_n,y_n), \gamma = f(x_n,y_n) - [x_n\frac{\partial f}{\partial x}(x_n,y_n) + y_n\frac{\partial f}{\partial y}(x_n,y_n)]$$

are constants.

Now we take the model problem $\frac{dy}{dx} = \lambda y$, where $\lambda = p + iq$ and $p \leq 0$. and solve it using our variant method as follows:

$$y_{n+1} = y_n + \frac{h}{2}(f(x_n + \frac{h}{4}, y(x_n) + \frac{h}{4}f_n) + f(x_n + \frac{3h}{4}, y(x_n) + \frac{3h}{4}f_n))$$
$$= y_n + \frac{h}{2}(f(x_n + \frac{h}{4}, y_n + \frac{h}{4}\lambda y_n) + f(x_n + \frac{3h}{4}, y_n) + \frac{3h}{4}\lambda y_n))$$
$$= y_n + \frac{h}{2}(\lambda(y_n + \frac{h}{4}\lambda y_n) + (y_n) + \frac{3h}{4}\lambda y_n))$$
$$= y_n + \frac{h}{2}(2\lambda y_n + h\lambda^2)y_n)$$
$$= y_n + (h\lambda y_n + \frac{h^2\lambda^2 y_n}{2})$$

$$= (1 + h\lambda + \frac{h^2\lambda^2}{2})y_n$$
$$= \sigma y_n \qquad (18)$$

where $\sigma = (1 + h\lambda + \frac{h^2\lambda^2}{2})$ Now for the stability of the method, we must have

$$|\sigma| \leq 1 \qquad (19)$$

i.e.

$$|1 + h\lambda + \frac{h^2\lambda^2}{2}| \leq 1 \qquad (20)$$

And we will consider the following equation to get the region of stability

$$|1 + h\lambda + \frac{h^2\lambda^2}{2}| = 1 = |exp(i\theta)| \qquad (21)$$

i.e.

$$(1 + h\lambda + \frac{h^2\lambda^2}{2}) - exp(i\theta) = 0 \qquad (22)$$

The above polynomial equation is solved for $0 \leq \theta \leq 2\pi$ the stability region is shown in the Figure-1.

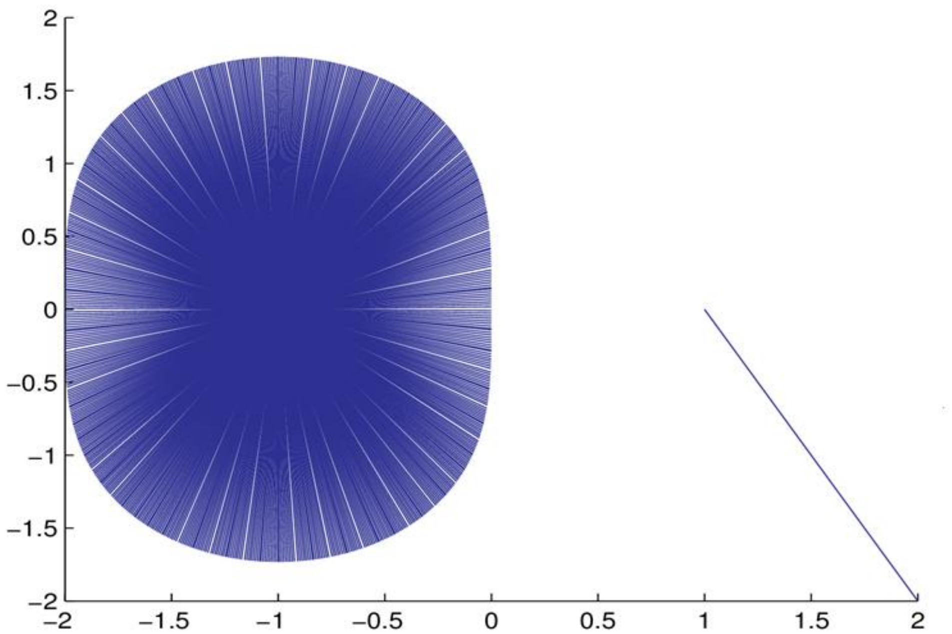

Figure 1: Stability region

It is observed from the figure-1 that the boundary of the stability region on the real axis is $|h\lambda_R| \leq 2$. This is very same as that of the 2nd order RK method. The stability region does not include the entire left half plane. So, this method is not A-stable.

5 Numerical examples

Problem-1:

$$\frac{dy}{dx} = 1 - \frac{y}{x}, y(2) = 2. \qquad (23)$$

Problem-2:

$$\frac{dy}{dx} = 1 - y, y(0) = 0. \qquad (24)$$

Problem-3:

$$\frac{dy}{dx} = \frac{x}{y}, y(0) = 0.5. \tag{25}$$

Problem-4:

$$\frac{dy}{dx} = -y, y(0) = 1. \tag{26}$$

Above four problems are solved numerically using the constructed variant method(VM) (12) . The exact solutions are evaluated analytically and the corresponding absolute errors are computed too. Finally the errors are compared with some of the same class of methods ,namely Euler(E),Modified Euler(ME),Improved Euler(IE), Improved modified Euler(IME), Modified improved modified Euler(MIME) and Runge-Kutta second order method(RK-2).And those are shown in Figure-2, Figure-3,Figure-4 and Figure-5 respectively. In the the above list of methods, Euler's method is an exception of being a first order method unlike other second order methods.

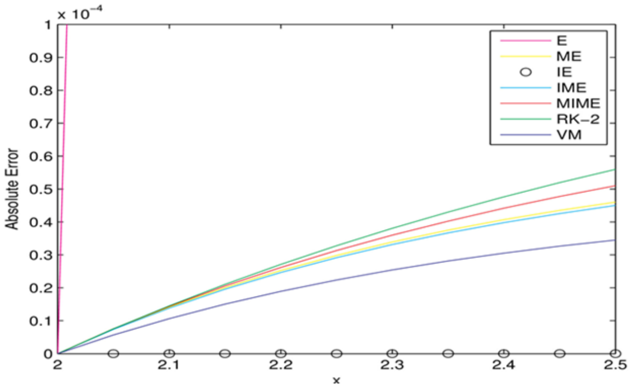

Figure 2: Error comparison for Problem-1

From the figure-2 it is found that the absolute error in the Problem-1 using the variant method is less than that of all other methods except that of the Improved Euler method (IE).

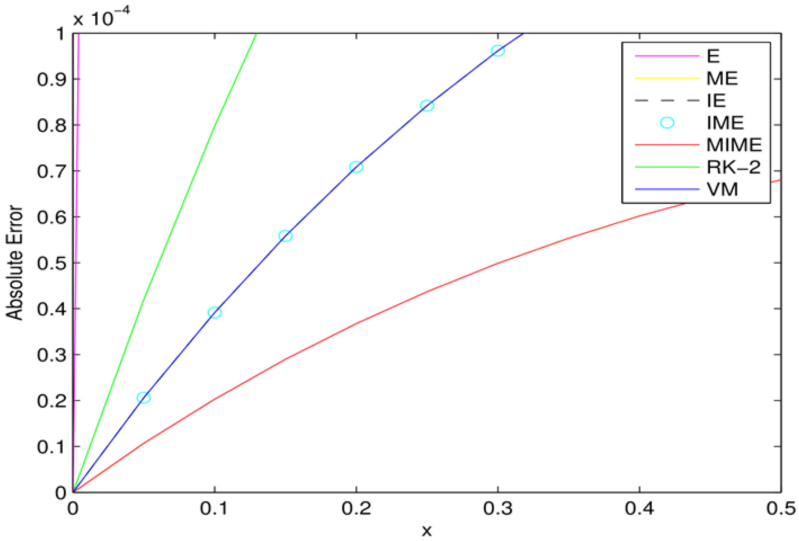

Figure 3: Error comparison for Problem-2

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From the figure-3 it is found that the absolute error in the Problem-2 using the variant method is less than or equal to that of all other methods except that of the Modified Improved Modified Euler method (MIME).

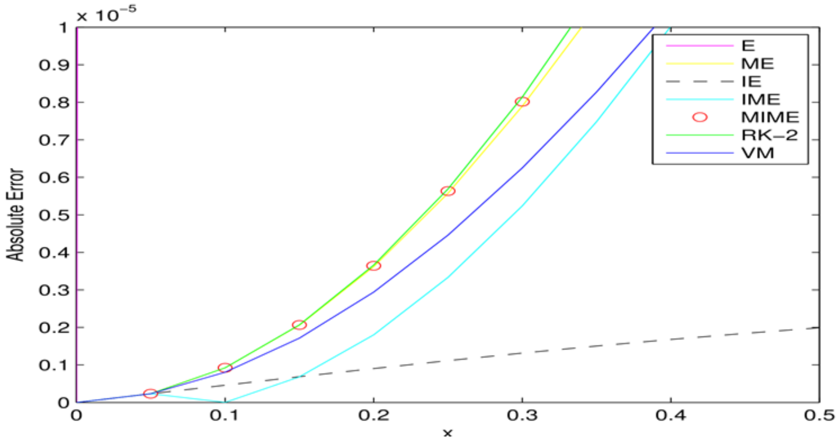

Figure 4: Error comparison for Problem-3

From the figure-4 it is found that the absolute error in the Problem-3 using the variant method is less than that of all other methods except that of the Improved Euler method (IE) and Improved Modified Euler Method (IME).

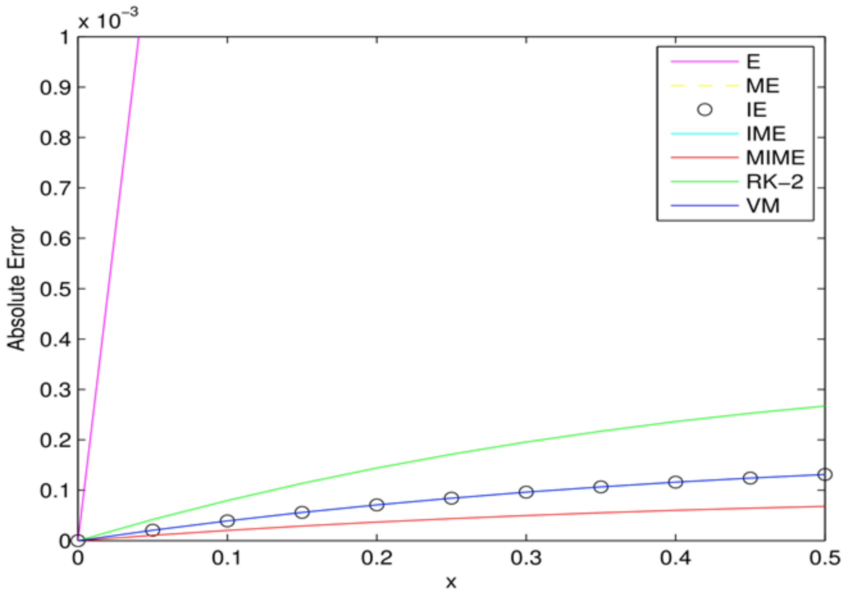

Figure 5: Error comparison for Problem-4

From the figure-5 it is found that the absolute error in the Problem-4 using the variant method is less than or e that of all other methods except that of the Modified Improved Euler method (MIME) .

6 Conclusion

In this paper, we have developed a new variant one step method to find the numerical solution of the Initial Value Problem in ordinary differential equation of all type. The error expression of the method is derived to ensure that the constructed method is of order 2.The stability stability regon of the Variant Method (VM) is studied and found to be the same as that of RK 2nd order method. Hence can be used as an alternative to RK 2nd order method. Numerical examples show that the Variant Method (VM) gives better result than most of the methods of the same class.This mehod gives a very good imprssion of becoming a handy tool to solve ordinary differential equations of all order.

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