TEST Engineering & Management

May-June 2020
ISSN: 0193-4120 Page No. 8959 - 8963

Aluthge Transformation on (α,β) normal operators

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Article Info
Volume 83
Page Number: 8959 - 8963
Publication Issue:
May - June 2020

Article History
***Article Received**: 19 November 2019*
***Revised**: 27 January 2020*
***Accepted**: 24 February 2020*
***Publication**: 18 May 2020*

Abstract:
In this article, we study polar decomposition and Aluthge transformation. We have derived and characterized the Aluthge transformation on a class of (α,β) Normal operators for $0 \leq \alpha \leq 1 \leq \beta$ and x ϵH.

AMS Subject Classification: 47B20,47B33.

***Keywords**: Aluthge transformation,*-Aluthge transformation, (α,β) Normal operators.*

1. Introduction

Theclasses of Hyponormal operators is generalized to the larger sets of p-hyponormal, log hyponormal, posinormal, etc., An operator D can be decomposed into $D=U\,|D|$ where U is partial isometry and $|D|$ is a square root of D^*D with N(U) = N($|D|$). In this article , we have studied new propertiesas an extension of hyponormal operators. This we have done using generalized Aluthge transform on (α,β) - normal operator.

Moreover, we prove that if $D=U\,|D|$ is (α,β) - normal operator, the Aluthuge transform is $\widetilde{D} = |D|^{\frac{1}{2}}U|D|^{\frac{1}{2}}$. For an operator $D=U\,|D|$ define $\widetilde{D}$ as follows:

$\widetilde{D}_{l,m} = |D|^{l}U|D|^{m}$ for l,m> 0 which is the generalized Aluthge transformation of D.

2. Preliminaries

Assume D as a bounded linear operator on a Hilbert space $\mathcal{H}$. Here, by anOperator, we mean a linear transformationbounded on a Hilbert space $\mathcal{H}$.

Definition
The bounded linear operator D on the Hilbert space H is a(α,β) - normal operator if
$\alpha^2(D^*D) \leq (DD^*) \leq \beta^2(D^*D), \quad 0 \leq \alpha \leq 1 \leq \beta.$

Proposition: (Douglas theorem)
For $A,B \in B(\mathcal{H})$ the following statements are similar:

a) $ran(A) \subseteq ran(B)$
b) $AA^* \leq \lambda^2 BB^*$ for some $\lambda > 0$ and
c) There existsa$D \in B(\mathcal{H})$ such that A = BD.

From the above results (a), (b) and (c) hold, then there is an unique operator D such that

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(1) $\|D\|^2 = \inf\{\mu : AA^* \le \mu BB^*\}$

(2) KerA = kerD

(3) $ranD \subseteq \overline{ranB^*}$

Using the above result, if D is (α,β) Normal operators we have

(i) $ranD = ran(D^*)$ equivalently $\ker D = \ker(D^*)$

(ii) There exists operators S_1, S_2 such that $D = D^* S_1$ and $D = S_2 D^*$

Furthermore S_1, S_2 are unique operators satisfying

(iii) $\|S_1\|^2 = \inf\{\beta \ge 1 : DD^* \le \beta D^* D\}$ and $\|S_2\|^2 = \sup\{\alpha > 0 : \alpha D^* D \le DD^*\}$ andker(D) = ker(S_1) = $\ker(S_2)$.

Proposition.[9]

Let A $\ge$ B $\ge$ 0. Then for all r > 0,

(1) $\left(D^{r/2} C^m D^{r/2}\right)^{1/n} \ge \left(D^{r/2} D^m D^{r/2}\right)^{1/n}$

(2) $\left(C^{r/2} C^m C^{r/2}\right)^{1/n} \ge \left(C^{r/2} D^m C^{r/2}\right)^{1/n}$

for m $\ge$ 0, n $\ge$ 1 with (1+r)n $\ge$ m + r.

The above inequality is called **Furuta Inequality.**

Proposition: McCarthy. [7]

Let B $\ge$ 0. Then

(i) $(Bx, x)^s \le \|x\|^{2(s-1)} (B^s x, x)$ if s $\ge$ 1.

(ii) $(Bx, x)^{sr} \ge \|x\|^{2(s-1)} (B^s x, x)$ if 0 $\le$ s $\le$ 1.

3. AluthgeTransformationon on (α,β) normal operators

Theorem 3.1: Let $D = U|D|$ be the polar decomposition of $(\alpha, \beta) - normaloperator$ then

$\tilde{D}_{l,m} = |D|^l U |D|^m \, is \, \frac{\text{mini}(l,m)}{l+m}$ - $(\alpha', \beta') - normaloperator$ for $l, m > 0$

Proof: Let D be $(\alpha, \beta) - normaloperator$ then

$$\alpha^2(D^*D) \le (DD^*) \le \beta^2(D^*D),$$

$$\alpha^2 |D|^2 \le |D^*|^2 \le \beta^2 |D|^2$$

Assume

$A = \alpha^2 |D|^2, B = |D^*|^2$ *and* $C = \beta^2 |D|^2$.Then

$$(\tilde{D}^*_{l,m} \tilde{D}_{l,m})^{\frac{\text{mini}(l,m)}{l+m}} = (|D|^m U^* |D|^{2l} U |D|^m)^{\frac{\text{mini}(l,m)}{l+m}}$$

$$= U^* (|D^*|^m |D|^{2l} |D^*|^m)^{\frac{\text{mini}(l,m)}{l+m}} U$$

$$= U^* (\beta^{-2l} B^{m/2} C^l B^{m/2})^{\frac{\text{mini}(l,m)}{l+m}} U$$

$$\ge \beta^{-2l \frac{\text{mini}(l,m)}{l+m}} U^* (B^{m/2} B^l B^{m/2})^{\frac{\text{mini}(l,m)}{l+m}} U$$

$$\ge \beta^{-2l \frac{\text{mini}(l,m)}{l+m}} |D|^{2\text{mini}(l,m)}$$

$$(\tilde{D}_{l,m} \tilde{D}_{l,m})^{\frac{\text{mini}(l,m)}{l+m}} = (|D|^m U^* |D|^{2l} U |D|^m)^{\frac{\text{mini}(l,m)}{l+m}}$$

$$=U^*(\left|D^*\right|^m \left|D\right|^{2l} \left|D^*\right|^m)^{\frac{\text{mini}(l,m)}{l+m}} U$$

$$=U^*(\alpha^{-2l} B^{m/2} A^l B^{m/2})^{\frac{\text{mini}(l,m)}{l+m}} U$$

$$\leq \alpha^{-2l\frac{\text{mini}(l,m)}{l+m}} U^*(B^{m/2} B^l v^{m/2})^{\frac{\text{mini}(l,m)}{l+m}} U$$

$$\leq \alpha^{-2l\frac{\text{mini}(l,m)}{l+m}} \left|D\right|^{2\text{mini}(l,m)}$$

Thus we have,

$$\left|D\right|^{2\text{mini}(l,m)} \geq \alpha^{2l} (\tilde{D}^*_{l,m} \tilde{D}_{l,m})^{\frac{\text{mini}(l,m)}{l+m}} \quad (1)$$

and

$$\left|D\right|^{2\text{mini}(l,m)} \leq \beta^{2l} (\tilde{D}^*_{l,m} \tilde{D}_{l,m})^{\frac{\text{mini}(l,m)}{l+m}} \quad (2)$$

$$(\tilde{D}_{l,m} \tilde{D}^*_{l,m})^{\frac{\text{mini}(l,m)}{l+m}} = (\left|D\right|^l U \left|D\right|^{2m} U^* \left|D\right|^l)^{\frac{\text{mini}(l,m)}{l+m}}$$

$$=\left(\left(\frac{C}{\beta^2}\right)^{l/2} \left|D^*\right|^{2m} \left(\frac{C}{\beta^2}\right)^{l/2}\right)^{\frac{\text{mini}(l,m)}{l+m}}$$

$$=\left(\left(\frac{C}{\beta^2}\right)^{l/2} B^m \left(\frac{C}{\beta^2}\right)^{l/2}\right)^{\frac{\text{mini}(l,m)}{l+m}}$$

$$\leq (\beta^2)^{\text{mini}(l,m)} (\tilde{D}^*_{l,m} \tilde{D}_{l,m})^{\frac{\text{mini}(l,m)}{l+m}}$$

$$(\tilde{D}_{l,m} \tilde{D}^*_{l,m})^{\frac{\text{mini}(l,m)}{l+m}} = (\left|D\right|^l U \left|D\right|^{2m} U^* \left|D\right|^l)^{\frac{\text{mini}(l,m)}{l+m}}$$

$$=\left(\left(\frac{A}{\alpha^2}\right)^{l/2} \left|D^*\right|^{2m} \left(\frac{A}{\alpha^2}\right)^{l/2}\right)^{\frac{\text{mini}(l,m)}{l+m}}$$

$$\geq \alpha^{-2l\frac{\text{mini}(l,m)}{l+m}} A^{\text{mini}(l,m)}$$

$$\geq (\alpha^2)^{\text{mini}(l,m)} (\tilde{D}^*_{l,m} \tilde{D}_{l,m})^{\frac{\text{mini}(l,m)}{l+m}}$$

So we have

$$(\tilde{D}_{l,m} \tilde{D}^*_{l,m})^{\frac{\text{mini}(l,m)}{l+m}} \leq (\beta^2)^{\text{mini}(l,m)} (\tilde{D}^*_{l,m} \tilde{D}_{l,m})^{\frac{\text{mini}(l,m)}{l+m}} \quad (3)$$

and

$$(\tilde{D}_{l,m} \tilde{D}^*_{l,m})^{\frac{\text{mini}(l,m)}{l+m}} \geq (\alpha^2)^{\text{mini}(l,m)} (\tilde{D}^*_{l,m} \tilde{D}_{l,m})^{\frac{\text{mini}(l,m)}{l+m}} \quad (4)$$

From (1), (2), (3) and (4),we have the result.

Since $\dfrac{l+m}{\text{mini}(l,m)} \geq 1$,

$\tilde{D}_{l,m}$ *is* $\dfrac{\text{mini}(l,m)}{l+m}$ $-(\alpha', \beta') - normaloperator$

Theorem 3.2: Assume $D = U\left|D\right|$ be the polar decomposition of $(\alpha, \beta) - normaloperator$ then $\tilde{D}_{l,m}$ *is* (α', β') – normal operator for $l, m > 0$

Proof:

Using Lowner-Heinz inequality, we get

$$\alpha^{2l}\left|D\right|^{2l} \leq \left|D^*\right|^{2l} \leq \beta^{2l}\left|D\right|^{2l}$$

If $0 < l, m \leq 1$, we have

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$$\widetilde{D}_{l,m}^{*}\widetilde{D}_{l,m} = |D|^m U^*|D|^{2l}U|D|^m$$

$$\geq \frac{|D|^m U^*|D|^{2l}U|D|^m}{\beta^{2l}}$$

$$\geq \beta^{-2l}|D|^m U^*|D|^{2l}U|D|^m$$

$$\geq \beta^{-2l}|D|^{2(l+m)}$$

$$\widetilde{D}_{l,m}^{*}\widetilde{D}_{l,m} \geq \beta^{-2l}|D|^{2(l+m)} \quad (5)$$

Similarly, we get

$$\widetilde{D}_{l,m}^{*}\widetilde{D}_{l,m} \leq \alpha^{-2l}|D|^{2(l+m)} \quad (6)$$

$$\widetilde{D}_{l,m}\widetilde{D}_{l,m}^{*} = |D|^l U|D|^{2m}U^*|D|^l$$

$$= |D|^l|D^*|^{2m}|D|^l$$

Since $|D^*| \geq |D|\alpha$, $|D^*| \leq \beta|D|$.

$$\widetilde{D}_{l,m}\widetilde{D}_{l,m}^{*} \leq \beta^{2m}|D|^{2(l+m)} \quad (7)$$

$$\widetilde{D}_{l,m}\widetilde{D}_{l,m}^{*} \geq \alpha^{2m}|D|^{2(l+m)} \quad (8)$$

From (5), (6), (7) and (8),we have the result.

Therefore $\widetilde{D}_{l,m}$ is $(\alpha',\beta')-$ normal operator.

Corollary 3.3: If $D = U|D|$ is $(\alpha,\beta) - normaloperator$,then $\widetilde{D}$ is also $(\alpha,\beta) - normaloperator$.

Theorem 3.4: Let $D = U|D|$ be $(\alpha,\beta) - normaloperator$,then

$$\beta^{-(n-1)}\left(D^*D\right) \leq \left(D^{n*}D^n\right)^{\frac{1}{n}} \leq \alpha^{-(n-1)}\left(D^*D\right)$$

holds$\forall$positive integer n.

Proof: Let $A_n = \left(D^{n*}D^n\right)^{\frac{1}{n}} = |D^n|^{\frac{2}{n}}$ and $B_n = \left(D^nD^{n*}\right)^{\frac{1}{n}} = |D^{n*}|^{\frac{2}{n}}$

By induction, $\beta^{-(n-1)}\left(D^*D\right) \leq \left(D^{n*}D^n\right)^{\frac{1}{n}} \leq \alpha^{-(n-1)}\left(D^*D\right)$ holds for $n = k$.

Since $A_k = \left(D^{*k}D^k\right)^{\frac{1}{k}} \geq \beta^{-(k-1)}\left(D^*D\right) \geq \beta^{-(k+1)}B_1$.

$$A_k = \left(D^{*k}D^k\right)^{\frac{1}{k}} \leq \alpha^{-(k-1)}\left(D^*D\right) \leq \alpha^{-(k+1)}B_1$$

It follows that,

$$\left(D^{(k+1)^*}D^{(k+1)}\right)^{\frac{1}{k+1}} = U^*\left(|D^*|D^{k^*}D^k|D^*|\right)^{\frac{1}{k+1}}U$$

$$= U^*\left(B_1^{\frac{1}{2}}A_k^kB_1^{\frac{1}{2}}\right)^{\frac{1}{k+1}}U$$

$$\geq \beta^{-k}(D^*D)$$

Similarly $\left(D^{(k+1)^*}D^{(k+1)}\right)^{\frac{1}{k+1}} \leq \alpha^{-k}(D^*D)$

Hence $\beta^{-(n-1)}\left(D^*D\right) \leq \left(D^{n*}D^n\right)^{\frac{1}{n}} \leq \alpha^{-(n-1)}\left(D^*D\right)$

Theorem 3.5: Let $D = U|D|$ be $(\alpha,\beta) - normaloperator$,then

$\beta^{-(n-1)}\left(D^nD^{n^*}\right)^{\frac{1}{n}} \leq \left(DD^*\right) \leq \alpha^{-(n-1)}\left(D^nD^{n^*}\right)^{\frac{1}{n}}$ holds for any positive integer n.

Proof: Let $A_n = \left(D^{n*}D^n\right)^{\frac{1}{n}} = |D^n|^{\frac{2}{n}}$ and $B_n = \left(D^nD^{n*}\right)^{\frac{1}{n}} = |D^{n*}|^{\frac{2}{n}}$ for any positive integer n

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Assume that

$\beta^{-(n-1)}\left(D^n D^{n^*}\right)^{\frac{1}{n}} \le \left(DD^*\right) \le \alpha^{-(n-1)}\left(D^n D^{n^*}\right)^{\frac{1}{n}}$ holds for $n = k$.

Then

$$A_1 = \left(D^* D\right) \ge \beta^{-2}\left(DD^*\right) \ge \beta^{-(k+1)}(D^k D^{k^*})^{\frac{1}{k}} \ge \beta^{-(k+1)} B_k$$

Similarly $A_1 \le \alpha^{-(k+1)} B_k$.

Hence we have

$$\left(D^{(k+1)} D^{(k+1)^*}\right)^{\frac{1}{k+1}} = U\,(\left|D^*\right| D^k D^{k^*} \left|D\right|)^{\frac{1}{k+1}} U$$

$$= U(A_1^{\frac{1}{2}} B_k^k A_1^{\frac{1}{2}})^{\frac{1}{k+1}} U^*$$

$$\le \beta^{\frac{k}{k+1}(k+1)} U\,(A_1^{\frac{1}{2}} A_1^k A_1^{\frac{1}{2}})^{\frac{1}{k+1}} U^*$$

byFuruta inequality

$$\le \beta^k \left|D^*\right|^2$$

Similarly $\left(D^{(k+1)} D^{(k+1)^*}\right)^{\frac{1}{k+1}} \ge \alpha^k \left|D^*\right|^2$

Therefore

$\beta^{-(n-1)}\left(D^n D^{n^*}\right)^{\frac{1}{n}} \le \left(DD^*\right) \le \alpha^{-(n-1)}\left(D^n D^{n^*}\right)^{\frac{1}{n}}$ holds for all positive integer n

Corollary 3.6: If D is $(\alpha,\beta) - normaloperator$ then D^n is $\frac{1}{n} - (\alpha^n, \beta^n) - normaloperator.$

Proof:Let D be $(\alpha,\beta) - normal\,operator$,then by theorems 3.4 and 3.5,we have

$$(D^{n^*} D^n)^{\frac{1}{n}} \ge \beta(D^* D) \ge \beta\beta^{-2}(DD^*) \ge \beta^{-2n}(D^n D^{n^*})^{\frac{1}{n}}$$

and

$$(D^{n^*} D^n)^{\frac{1}{n}} \le \alpha^{-2n}(D^n D^{n^*})^{\frac{1}{n}}$$

Hence D^n is $\frac{1}{n} - (\alpha^n, \beta^n) - normaloperator.$

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