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A New Approach of Fuzzy Brownian Motion

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Abstract

The Brownian motion concept is an efficient tool to handle the uncertain systems. But in case of hybrid uncertain systems(Randomness and Fuzziness), Brownian motion is not sufficient. To handle such situation, we extend this tool in fuzzy environment. In this paper the properties of fuzzy Brownian motion are derived which satisfy the fuzzy Gaussian process and fuzzy Markov property. Also the quadratic variation and fuzzy martingale of fuzzy Brownian motion are discussed.

Keywords: *Fuzzy Brownian motion(FBM); Fuzzy Gaussian process(FGP); Quadratic variation(QV) of fuzzy Brownian motion (FBM); Fuzzy martingale(FM); Fuzzy Markov property(FMP); Moment generating function(MGF).*

I. Introduction:-

The concept of stochastic process(SP) was first derived by Joseph Doob which is a random function of time i.e. $\{X(t), t \in [0,T]\}$. Stochastic process(SP) takes an prominent role in a range of application areas such as physics, chemistry, biology, microelectronics, finance etc.

Brownian motion (BM) describes the location of particle at time t i.e. $\{B(t), t \in [0,t]\}$. An English botanist Robert Brown observed the motion of polen particle suspended in a fluid. After that Bachelier used the BM in stock market[1], [2]. Then Einestein obtained the equation of BM for SP. The mathematical formulation of BM as a SP was given by Winner.

Two mathematician has discussed a prototypical which tells nature prices of stock which used BM[2], [3]. After that, many researchers developed different model or strategies using BM in SP. Auria and Kella describe the Markov process(MP) behaves like a two sided reflected BM in fluid [4]. Then Dorogovtsev and Lzyumtsevaparadigm the self connection local time fordemonstrate image of the planner BM.

Fuzzy set theory was introduced by Zadeh also, he introduced the concept of fuzzy random variable. After that, many researchers have developed fuzzy theory in different branches of mathematics [5]. Kwakernaak, Puri and Ralesce have been developed the variables of fuzzy random whose parameters are fuzzy numbers. Puri and Ralesce have developed the concept of convergence theorem of FM. Also, they introduced the concept of fuzzy Gaussian process(FGP). Further, Stojakovic discussed FM [6]. Consequently FRV, fuzzy expectations, FM etc. have been discussed by Li and Ogura. Also, they developed FGP and FM property . Using Li and Ogura’s concept Li and Ren described FGP and FBM .

Researchers discussed a nonlinear fuzzy chance constrained programing problem. Buckley developed Fuzzy Markov chain[7], [8]. Wang and Zhang introduced fuzzy stochastic process(FSP). Kim and Kim developed fuzzy stochastic differential equation which includes a fuzzy set valued FBM diffusion process. Researcherdiscussed the fuzzy set valued BM and GP and proved their various properties. Further, Bongiorno describe the properties of fuzzy set valued BM which based on Li and Ogura’s results. Babiszewska derived fuzzy modeling stochastic process by using BM. Piriyakumar and Sreevinotha discussed the fuzzy Markov chain(FMC) by using the transition probability. Moreover 2016 Ryan Case derived FMC.

In the present study some work has been done on FBM. Many researchers have discussed the BM as a function of crisp time[9], [10]. But in general time may not be fixed or crisp rather there may be uncertain. In this paper, we considered the

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uncertain time as fuzzy time.

This paper is sequenced as follows: section-2 contains few definations on BM and FBMs. In section-3 we define the properties FBM and proved some important results by using Buckley's method. the FBM as a FM is discussed in Section-4. Finally the conclusion and concluding remarks are made in section-5 followed by supporting reference.

2 Basic Priliminaries:-

2.1 Fuzzy Random variable(FRV):-

"Fuzzy random variable is known as a random variable with parameters such as mean, variance etc. are fuzzy numbers".

2.2 Fuzzy number(FN):-

"A fuzzy number $\tilde{a}$ is known as a convex standardized fuzzy set $\tilde{a}$ of real line IR having membership function $\mu(\tilde{a})$:R →[0,1.0] that satisfy the following conditions.

(i) one interval exists exactly I $\in R$, It may be singleton, such that $\mu(\tilde{a}(x)) = 1.0$ for all $x \in I$.

(ii)" $\mu(\tilde{a}(x))$ is a piecewise continuos of membership function".

2.3 The α-cut:-

"Let $\tilde{u} \in F(R)$ be a fuzzy number, the α-cut of $\tilde{u}$, for every $\alpha \in [0,1]$, the set

$$[\tilde{u}]_\alpha = \{x \in R: \tilde{u}(x) \geq \alpha\} = [\tilde{u}_\alpha^L, \tilde{u}_\alpha^U]$$

$$where[\tilde{u}]_\alpha^L = \inf_{x \in \mathrm{I}R}\{x \in [\tilde{u}]_\alpha\} and [\tilde{u}]_\alpha^U = \sup_{x \in \mathrm{I}R}\{x \in [\tilde{u}]_\alpha\},$$

the support of $\tilde{u}$ is given by $[\tilde{u}]_0 = Support(\tilde{u}) = \{x \in \mathrm{I}R: \tilde{u}(x) > 0\}$.

2.4 Inequalities:-

It follows "It is a partial relationship of order between two fuzzy numbers. Researchers have defined a partial order relation $\preccurlyeq$ and $\prec$ between two fuzzy numbers $\tilde{a}$ and $\tilde{b}$ with α-cuts $\tilde{a}[\alpha]$ and $\tilde{b}[\alpha]$ respectively as "$\tilde{a} \succcurlyeq \tilde{b}$" iff "$a > b$" for "$a \in \tilde{a}[\alpha]$" and "$b \in \tilde{b}[\alpha]$" for each $\alpha \in [0,1]$. This definition of $\tilde{a} \succcurlyeq \tilde{b}$ is equivalent to the following form.

Let $\tilde{a}[\alpha] = [a_*, a^*]$ and $\tilde{b}[\alpha] = [b_*, b^*]$.

So $\tilde{a} \succcurlyeq \tilde{b}$ iff $[a_* \geq b^*]$ and $\tilde{a} \preccurlyeq \tilde{b}$ iff $[a^* \leq b_*]$ for each $\alpha \in [0,1]$.

The benefit of this category of partial order relation is that it can reduces mathematical computation".

2.5 Fuzzy martiangle (FM):-

It follows (9) "The sequence $\{\tilde{X}_n, \tilde{\mathcal{F}}_n\}_n$ of variables of fuzzy random and σ- algebra is a fuzzy martiangle at n $\geq$ 1,

(i) $\tilde{X}_n is \tilde{\mathcal{F}}_n$ measurable and $E \parallel supp\tilde{X}_n \parallel < \infty$.

(ii) $E(\tilde{X}_{n+1}|\tilde{\mathcal{F}}_n) = \tilde{X}_n$.

If properties (ii) is replaced by

(iii) $E(\tilde{X}_{n+1}|\tilde{\mathcal{F}}_n) \geq \tilde{X}_n$, $(E(\tilde{X}_{n+1}|\tilde{\mathcal{F}}_n) \leq \tilde{X}_n)$,

then $\{\tilde{X}_n, \tilde{\mathcal{F}}_n\}_n$ is called fuzzy sub-martingale (super-martingale) respectively.

2.6 Fuzzy stochastic process(FSP):-

It follows "A fuzzy family $\{\tilde{X}(t), t \in [0,T], 0 < T < \infty\}$ is known as a fuzzy stochastic process of probability space (Ω, A, P) only if $\forall\alpha \in [0,1]$

$$\tilde{X}_\alpha(t,\omega) = [\tilde{X}_\alpha^L(t,\omega), \tilde{X}_\alpha^U(t,\omega)],$$

and the process is a process of stochastic process (SP) on (Ω, A, P) and

$$\tilde{X}(t,\omega) = \cup_{\alpha\in[0,1]} \tilde{X}_\alpha(t,\omega).$$

2.7 The interval valued BM:-

It follows "A family $\{\overline{B}(t), t \in [0,T], 0 < T < \infty\}$ is an Brownian motion of interval valued on probability space (Ω, A, P) "

$$\overline{\mathcal{B}}(t,\omega) = [\overline{\mathcal{B}}^L(t,\omega), \overline{\mathcal{B}}^U(t,\omega)],$$

where $\overline{\mathcal{B}}^L(t,\omega) and \overline{\mathcal{B}}^U(t,\omega)$ are Brownian motions such that

$$P[\overline{\mathcal{B}}^L(t,\omega) \leq \overline{\mathcal{B}}^U(t,\omega)] = 1.$$

2.8 Fuzzy Brownian Motion(FBM):-

This follows as "A Fuzzy stochastic process $\{\tilde{B}(t), t \in [0,T], 0 < T < \infty\}$ is called a fuzzy Brownian motion on probability space (Ω, A, P) if and only if $\forall\alpha \in [0,1]$, the process

$$\tilde{\mathcal{B}}_\alpha(t,\omega) = [\tilde{\mathcal{B}}_\alpha^L(t,\omega), \tilde{\mathcal{B}}_\alpha^U(t,\omega)]$$

is an interval Brownian motion on (Ω, A, P) and

$$\tilde{\mathcal{B}}(t,\omega) = \cup_{\alpha\in[0,1]} \tilde{\mathcal{B}}_\alpha(t,\omega).$$

2.9 Fuzzy Gaussian process(FGP):-

It follows "A process of fuzzy stochastic $\{\tilde{X}(t), t \in [0,T], 0 \prec T \prec \infty\}$ known as FGP on the probability space (Ω, A, P) if and only if $\forall\alpha \in [0,1.0]$, the family $\{\tilde{X}_\alpha(t), t \in [0,T], 0 \prec T \prec \infty\}$ is known as a process of interval valued Gaussian.

2.10 Quadratic variation(QV) of BM:-

It follows "The variation of quadratic of BM $[\mathcal{B},\mathcal{B}](t)$" is defined as

$$[\mathcal{B},\mathcal{B}](t) = [\mathcal{B},\mathcal{B}]([0,t]) = \lim_{n\to\infty} \sum_{i=1}^n |\mathcal{B}(t_i) - \mathcal{B}(t_{i-1})|^2.$$

2.11 Quadratic variation(QV) of FBM:-

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The QV of FBM $\widetilde{\mathcal{B}}(t)$ over $[\tilde{0},\tilde{t}]$ is $[\widetilde{\mathcal{B}}_\alpha,\widetilde{\mathcal{B}}_\alpha](t)$, $\forall\alpha\in[0,1]$

$$=\{\lim_{n\to\infty}\sum_{i=1}^{n}|\mathcal{B}(t_i)\ominus\mathcal{B}(t_{i-1})|^2|\mathcal{B}(t_i)\in\widetilde{\mathcal{B}}_\alpha(t_i),\mathcal{B}(t_{i-1})\in\widetilde{\mathcal{B}}_\alpha(t_{i-1})\}.$$

2.12 Moment generating function(MGF) of BM:-

"The generating function of moment of a BM is:

$$M_{\mathcal{B}(t)}(u)=E\left[e^{u\mathcal{B}(t)}\right]=e^{\frac{u^2}{2}t},\ u\in\mathrm{R}$$

2.13 MGF of FBM:-

For $\alpha\in[0,1]$, the MGF of FBM $\widetilde{\mathcal{B}}(t)\sim N(\tilde{0},\tilde{t})$ is

$$\{M_{\mathcal{B}(t)}(\lambda)|\mathcal{B}(t)\in\widetilde{\mathcal{B}}_\alpha(t)\}=\{E[e^{\lambda\mathcal{B}(t)}]|\mathcal{B}(t)\in\widetilde{\mathcal{B}}_\alpha(t)\}=\{e^{\frac{\lambda^2}{2}t}|t\in\tilde{t}_\alpha\},(\lambda\in R)$$

2.14 Markov property(MP):-

"$X(t)$ is a process of Markov if for any t and s >0, the conditional delivery of X(t+s) given $\mathcal{F}_t$ issimilar as conditional distribution of $X(t)$stated$X(t)$, that is,

$$P(X(t+s)\le y|\mathcal{F}_t)=P(X(t+s)\le y|\mathcal{B}(t))$$

2.15 Fuzzy Markov property(FMP):-

$\tilde{X}(t)$ is a FMP if any values of $\tilde{t},\tilde{s}\succ\tilde{0}$, the fuzzy conditional distribution of $\tilde{X}_\alpha(t+s),\alpha\in[0,1]$ given FM $\tilde{\mathcal{F}}_t$ is equal to the fuzzy conditional distribution of $\tilde{X}(t+s)$ given $\widetilde{\mathcal{B}}(t)$ i.e. the is α-cut of FMP is

$$\{P(X(t+s)|\tilde{\mathcal{F}}_t)|X(t+s)\in\tilde{X}_\alpha(t+s)\}=\{P(X(t+s)|\mathcal{B}(t))|X(t+s)\in\tilde{X}_\alpha(t+s),\mathcal{B}(t)\in\widetilde{\mathcal{B}}_\alpha$$

3 Fuzzy Brownian motion(FBM):-

In order to derive the quadratic variation of FBM, it is required to define the properties of FBM as follows.

(i) In any fuzzy interval $[\tilde{0},\tilde{t}]$, FBM is a continuous function of $\tilde{t}$.

(ii) Any fuzzy interval over $[\tilde{0},\tilde{t}]$, FBM are not monotone.

(iii)Any fuzzy interval, the variation of FBM is infinity.

(iv) FBM satisfies independent increment property i.e for $\tilde{t}\succ\tilde{s}$, $\widetilde{\mathcal{B}}(t)\ominus\widetilde{\mathcal{B}}(s)$ is not depend of past i.e. $\widetilde{\mathcal{B}}(s)$.

(v)FBM satisfies fuzzy normal increment property i.e.

$$\forall\alpha\in[0,1]\ \{\widetilde{\mathcal{B}}(t)\ominus\widetilde{\mathcal{B}}(s)|\mathcal{B}(t)\in\widetilde{\mathcal{B}}_\alpha(t),\mathcal{B}(s)\in\widetilde{\mathcal{B}}_\alpha(t)\}$$

has fuzzy normal dispersal with mean $\tilde{0}$ and adjustment is

$$\{\tilde{t}\ominus\tilde{s}|t\in\tilde{t}_\alpha,s\in\tilde{s}_\alpha\}.$$

Here the end points of each of partions are supposed as fuzzy numbers.

Theorem 3.1

QV of FBM $\widetilde{\mathcal{B}}(t)$ over $[\tilde{0},\tilde{t}]$ is $\tilde{t}$.

Proof:-

$\widetilde{\mathcal{B}}(t)$ be a FBM.

Let $\Delta\mathcal{B}=\{\mathcal{B}(t_i)|\mathcal{B}(t_i)\in\widetilde{\mathcal{B}}_\alpha(t_i),\mathcal{B}(t_{i-1})|\mathcal{B}(t_{i-1})\in\widetilde{\mathcal{B}}(t)_\alpha(t_{i-1})\}$,

$\Delta t=\{t_i|t_i\in(\tilde{t}_i)_\alpha,t_{i-1}|t_{i-1}\in(\tilde{t}_{i-1})_\alpha\}$.

For each $\alpha\in[0,1]$, $\widetilde{\mathcal{B}}_\alpha(t)$ is a an interval valued BM.

$\widetilde{\mathcal{B}}_\alpha(t)=[\widetilde{\mathcal{B}}_\alpha^U(t),\widetilde{\mathcal{B}}_\alpha^L(t)]$ is the α-cut of the FBM $\widetilde{\mathcal{B}}(t)$.

The QV of FBM $\widetilde{\mathcal{B}}(t)$ over $[\tilde{0},\tilde{t}]$ is

$$[\widetilde{\mathcal{B}},\widetilde{\mathcal{B}}](t)=\lim_{n\to\infty}\sum_{i=1}^{n}|\widetilde{\mathcal{B}}(t_i)\ominus\widetilde{\mathcal{B}}(t_{i-1})|^2.\tag{1}$$

$$Let\tilde{Z}=\sum_{i=1}^{n}|\widetilde{\mathcal{B}}(t_i)\ominus\widetilde{\mathcal{B}}(t_{i-1})|^2$$

As $\widetilde{\mathcal{B}}(t)$ is a FBM so $\tilde{Z}$ is a FRV.

The α-cut of $\tilde{Z}$ is defined as $\tilde{Z}_\alpha=[\tilde{Z}_\alpha^U,\tilde{Z}_\alpha^L]\forall\in[0,1]$.

The α-cut of expectation of $\tilde{Z}$ is

$$(E[\tilde{Z}])_\alpha=\{\tilde{E}[\textstyle\sum_{i=1}^{n}|\mathcal{B}(t_i)\ominus\mathcal{B}(t_{i-1})|^2]|\Delta\mathcal{B}\}$$

$$=\{\textstyle\sum_{i=1}^{n}E|\mathcal{B}(t_i)\ominus\mathcal{B}(t_{i-1})|^2|\Delta\mathcal{B}\}$$

$$=\{\textstyle\sum_{i=1}^{n}[Var(\mathcal{B}(t_i)\ominus\mathcal{B}(t_{i-1}))]|\Delta\mathcal{B}\}$$

$$=\{\textstyle\sum_{i=1}^{n}[Var(\mathcal{B}(t_i)\oplus Var(\mathcal{B}(t_{i-1})\ominus 2Cov(\mathcal{B}(t_i),\mathcal{B}(t_{i-1}))]|\Delta\mathcal{B}\}$$

$$=\{\textstyle\sum_{i=1}^{n}[t_i\oplus t_{i-1}\ominus 2min(t_{i-1},t_i)]|\Delta t\}$$

$$=\{\textstyle\sum_{i=1}^{n}(t_i\oplus t_{i-1}\ominus 2t_{i-1})|\Delta t\}$$

$$=\{\textstyle\sum_{i=1}^{n}(t_i\ominus t_{i-1})|\Delta t\}=\tilde{t}_\alpha$$

$$(\tilde{E}(\tilde{Z}))_\alpha=\tilde{t}_\alpha,\forall\alpha\in[0,1].\tag{2}$$

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To prove the QV of FBM $\tilde{\mathcal{B}}(t)$ over $[\tilde{0},\tilde{t}]$ is $\tilde{t}$.

i.e. to show $\tilde{Z}_\alpha \ominus (E[\tilde{Z}])_\alpha \to \tilde{0}$ in fuzzy probability, $\forall \in [0,1]$.

Now the α-cut of the variance of $\tilde{Z}$ is

$$(Var[\tilde{Z}])_\alpha = \{Var(\sum_{i=1}^{n} |\mathcal{B}(t_i) \ominus \mathcal{B}(t_{i-1})|^2)| \vartriangle \mathcal{B}\}$$

$$= \{\sum_{i=1}^{n} Var|\mathcal{B}(t_i) \ominus \mathcal{B}(t_{i-1})|^2| \vartriangle \mathcal{B}\}$$

$$= \{\sum_{i=1}^{n} 3(t_i \ominus t_{i-1})^2| \vartriangle t\}$$

$$\preccurlyeq \{3max(t_i \ominus t_{i-1})\sum_{i=1}^{n} (t_i \ominus t_{i-1})| \vartriangle t\}$$

$$= \{3\delta_n t | t \in \tilde{t}_\alpha\}, \forall\alpha \in [0,1]. As(\delta_n)_\alpha = \{max(t_i \ominus t_{i-1})| \vartriangle t\},$$

So, for each $\alpha \in [0,1]$, $\lim_{n\to\infty}(Var[Z])_\alpha = \tilde{0}$

$$\lim_{n\to\infty}(E[Z \ominus E[Z]])_\alpha = \tilde{0}$$

Since $\tilde{Z}_\alpha$ converges to $(\tilde{E}[\tilde{Z}])_\alpha$ $\forall\alpha \in [0,1]$

Hence $\tilde{Z}_\alpha \ominus (\tilde{E}[\tilde{Z}])_\alpha \to \tilde{0}$.

Theorem 3.2

A FBM $\tilde{\mathcal{B}}(t)$ is a FGP iff with the FGP is mean $\tilde{0}$ and covariance is $min(\tilde{t},\tilde{s})$.

Proof:- Let $\tilde{\mathcal{B}}(t)$ and $\tilde{\mathcal{B}}(t)(s)$ are two FBM.

Let $\vartriangle \mathcal{B}_1 = \{\mathcal{B}(t)|\mathcal{B}(t) \in \tilde{\mathcal{B}}_\alpha(t)\}$, $\vartriangle \mathcal{B}_2 = \{\mathcal{B}(s)|\mathcal{B}(s) \in \tilde{\mathcal{B}}_\alpha(s)\}$,

$\vartriangle \mathcal{B}_3 = \{\mathcal{B}(t+s)|\mathcal{B}(t+s) \in \tilde{\mathcal{B}}_\alpha(t+s)\}$

$\vartriangle t = \{t|t \in \tilde{t}_\alpha\}$, $\quad \vartriangle s = \{s|t \in \tilde{s}_\alpha\}$

For each $\alpha \in [0,1]$, $\tilde{\mathcal{B}}_\alpha(t)$ and $\tilde{\mathcal{B}}_\alpha(s)$ and $\tilde{\mathcal{B}}_\alpha(t+s)$ are the interval valued BMs.

$\tilde{\mathcal{B}}_\alpha(t) = [\tilde{\mathcal{B}}_\alpha^U(t), \tilde{\mathcal{B}}_\alpha^L(s)]$ and $\tilde{\mathcal{B}}_\alpha(s) = [\tilde{\mathcal{B}}_\alpha^U(s), \tilde{\mathcal{B}}_\alpha^L(s)]$ are FBMS.

The mean of FBM is $\tilde{0}$.

$$\{E[\mathcal{B}(t)]| \vartriangle \mathcal{B}_1\} = \tilde{0} and \{E[\mathcal{B}(s)]| \vartriangle \mathcal{B}_2\} = \tilde{0} \qquad (3)$$

To prove the covariance function of $\tilde{\mathcal{B}}(t)$ and $\tilde{\mathcal{B}}(s)$ is min $(\tilde{t},\tilde{s})$.

i.e. $\{Cov\mathcal{B}(t),\mathcal{B}(s)| \vartriangle \mathcal{B}_1, \vartriangle \mathcal{B}_2\} = \{min(t,s)| \vartriangle t, \vartriangle s\}$.

Now the α-cut of covariance of FBMs are $\{Cov(\mathcal{B}(t),\mathcal{B}(s))| \vartriangle \mathcal{B}_1, \vartriangle \mathcal{B}_2\}$

$$= \{E[(\mathcal{B}(t) \ominus E(\mathcal{B}(t)))(\mathcal{B}(s) \ominus E(\mathcal{B}(s)))]| \vartriangle \mathcal{B}_1, \vartriangle \mathcal{B}_2\}$$

$$= \{E[\mathcal{B}(t) \otimes \mathcal{B}(S)]| \vartriangle \mathcal{B}_1, \vartriangle \mathcal{B}_2\}$$

$$[Cov(\mathcal{B}(t),\mathcal{B}(s))]_\alpha = [E(\mathcal{B}(t) \otimes \mathcal{B}(s))]_\alpha$$

Takeing $\tilde{t} \succ \tilde{s}$ in $\tilde{\mathcal{B}}(t)$.

$$\tilde{\mathcal{B}}(t) = \tilde{\mathcal{B}}(s) \oplus \tilde{\mathcal{B}}(t) \ominus \tilde{\mathcal{B}}(s).$$

$$\tilde{\mathcal{B}}(t) \otimes \tilde{\mathcal{B}}(s) = (\tilde{\mathcal{B}}^2(s)) \oplus [\tilde{\mathcal{B}}(s) \otimes (\tilde{\mathcal{B}}(t) \ominus \tilde{\mathcal{B}}(s))]$$

Now the α-cut of the expectation of $\tilde{\mathcal{B}}(t) \otimes \tilde{\mathcal{B}}(s)$ is

$$\{E[\mathcal{B}(t) \otimes \mathcal{B}(s)]| \vartriangle \mathcal{B}_1, \vartriangle \mathcal{B}_2\} = \{E[\mathcal{B}^2(s) \oplus (\mathcal{B}(s) \otimes (\mathcal{B}(t) \ominus \mathcal{B}(s))]| \vartriangle \mathcal{B}_1, \vartriangle \mathcal{B}_2\}$$

$$= \{E[\mathcal{B}^2(s)]| \vartriangle \mathcal{B}_2\} \oplus \{E[\mathcal{B}(s) \otimes (\mathcal{B}(t) \ominus \mathcal{B}(s))]| \vartriangle \mathcal{B}_1, \vartriangle \mathcal{B}_2\}$$

$$= \{s|s \in \tilde{s}_\alpha\} \oplus E[\mathcal{B}(s)] \otimes E[\mathcal{B}(t) \ominus \mathcal{B}(s)]| \vartriangle \mathcal{B}_1, \vartriangle \mathcal{B}_2\} = \tilde{s}_\alpha (By(3))$$

For each $\alpha \in [0,1]$, $\{Cov(\mathcal{B}(t),\mathcal{B}(s)| \vartriangle \mathcal{B}_1, \vartriangle \mathcal{B}_2)\} = \tilde{s}_\alpha$

Similarly if $\tilde{t} \prec \tilde{s}$, $\{Cov(\mathcal{B}(t),\mathcal{B}(s))| \vartriangle \mathcal{B}_1, \vartriangle \mathcal{B}_2\} = \tilde{t}_\alpha$

Hence $\{Cov(\mathcal{B}(t),\mathcal{B}(s))| \vartriangle \mathcal{B}_1, \vartriangle \mathcal{B}_2\} = \{min(t,s)| \vartriangle t, \vartriangle s\}$

Conversely, given $\tilde{\mathcal{B}}(t)$ be a FGP with mean $\tilde{0}$ and covariance is $min(\tilde{t},\tilde{s})$

The joint fuzzy probability distribution of $\tilde{\mathcal{B}}(t)$ and $\tilde{\mathcal{B}}(t+s)$ is a bivariate fuzzy normal distribution with mean $\tilde{0}$.

$\tilde{\mathcal{B}}(t)$ and $\tilde{\mathcal{B}}(t+s) \ominus \tilde{\mathcal{B}}(t)$ are also bivariate variate normal distribution.

As $\{Cov(\mathcal{B}(t),\mathcal{B}(s))| \vartriangle \mathcal{B}_1, \vartriangle \mathcal{B}_2\} = \{min(t,s)| \vartriangle t, \vartriangle s\}$

$\{Cov(\mathcal{B}(t),\mathcal{B}(t+s))| \vartriangle \mathcal{B}_1, \vartriangle \mathcal{B}_3\} = \{min(t+s,t)| \vartriangle t, \vartriangle s\}$

The α-cut of covariance of $\tilde{\mathcal{B}}(t)$ and $\tilde{\mathcal{B}}(t+s) - \tilde{\mathcal{B}}(t)$ is

$$\{Cov(\mathcal{B}(t),\mathcal{B}(t+s) \ominus \mathcal{B}(t))| \vartriangle \mathcal{B}_1, \vartriangle \mathcal{B}_3\}$$

$$\{Cov(\mathcal{B}(t),\mathcal{B}(t+s))| \vartriangle \mathcal{B}_1, \vartriangle \mathcal{B}_3\} \ominus \{Cov(\mathcal{B}(t),\mathcal{B}(t))| \vartriangle \mathcal{B}_1\}$$

$$= \{t| \vartriangle t\} \ominus \{t| \vartriangle t\} = \tilde{t}_\alpha - \tilde{t}_\alpha = \tilde{0}$$

$$\{Cov(\mathcal{B}(t),\mathcal{B}(t+s) \ominus \mathcal{B}(t))| \vartriangle \mathcal{B}_1, \vartriangle \mathcal{B}_3\} = \tilde{0}$$

If covariance of two FRV are zero then they are independent.

$\tilde{\mathcal{B}}(t)$ and $\tilde{\mathcal{B}}(t+s) \ominus \tilde{\mathcal{B}}(t)$ are not dependent.

The increment $\tilde{\mathcal{B}}(t+s) \ominus \tilde{\mathcal{B}}(t)$ is not depend on $\tilde{\mathcal{B}}(t)$.

Hence $\tilde{\mathcal{B}}(t)$ be a FBM.

Theorem 3.3

FBM $\tilde{\mathcal{B}}(t)$ holds the FMP

Proof:- Let $\vartriangle \mathcal{B}_1 = \{\mathcal{B}(t)|\mathcal{B}(t) \in \tilde{\mathcal{B}}_\alpha(t)\}$, $\vartriangle \mathcal{B}_2 = \{\mathcal{B}(t \oplus s)|\mathcal{B}(t+s) \in \tilde{\mathcal{B}}_\alpha(t+s)\}$

Let for each $\alpha \in [0,1]$, $\tilde{\mathcal{B}}_\alpha(t)$ and $\tilde{\mathcal{B}}_\alpha(t+s)$ are two interval valued BMs.

$\tilde{\mathcal{B}}_\alpha(t) = [\tilde{\mathcal{B}}_\alpha^U(t), \tilde{\mathcal{B}}_\alpha^L(s)]$ and $\tilde{\mathcal{B}}_\alpha(t \oplus s) = [\tilde{\mathcal{B}}_\alpha^U(t+s), \tilde{\mathcal{B}}_\alpha^L(t+s)]$ be the α-cuts of FBMs $\tilde{\mathcal{B}}(t)$ and $\tilde{\mathcal{B}}(t+s)$ respectively.

To prove FBM satisfies FMP, Using moment generating function of FBM it is equivalent to the conditional distribution of $\tilde{\mathcal{B}}(t+s)$ given fuzzy martingale $\tilde{\mathcal{F}}_t$ is same as FBM $\tilde{\mathcal{B}}(t)$.

Now $\{e^{u\mathcal{B}(t+s)}| \vartriangle \mathcal{B}_2\} = \{e^{u(\mathcal{B}(t) \oplus (\mathcal{B}(t+s) \ominus \mathcal{B}(t)))}| \vartriangle \mathcal{B}_1, \vartriangle \mathcal{B}_2\}$.

As the MGF of FBM $\tilde{\mathcal{B}}(t+s)$ is the expectation of $e^{u\tilde{\mathcal{B}}(t+s)}$.

So $M_{\tilde{\mathcal{B}}_\alpha(t+s)}(u) = \{e^{u\mathcal{B}(t)} \otimes e^{u(\mathcal{B}(t+s) \ominus \mathcal{B}(t))}| \vartriangle \mathcal{B}_1, \vartriangle \mathcal{B}_2\}$

$$\{E[e^{u\mathcal{B}(t+s)}|\tilde{\mathcal{F}}_t]| \vartriangle \mathcal{B}_2\} = \{E[e^{u(\mathcal{B}(t)+(\mathcal{B}(t+s) \ominus \mathcal{B}(t)))}|\tilde{\mathcal{F}}_t]| \vartriangle \mathcal{B}_1, \vartriangle \mathcal{B}_2\}$$

$$= \{E[e^{u\mathcal{B}(t)}|\tilde{\mathcal{F}}_t]| \vartriangle \mathcal{B}_1\} \otimes \{E[e^{u(\mathcal{B}(t+s) \ominus \mathcal{B}(t))}|\tilde{\mathcal{F}}_t]| \vartriangle \mathcal{B}_1, \vartriangle \mathcal{B}_2\}$$

$$= \{e^{u\mathcal{B}(t)}| \vartriangle \mathcal{B}_1\} \otimes \{E[e^{u(\mathcal{B}(t+s) \ominus \mathcal{B}(t))}]| \vartriangle \mathcal{B}_1, \vartriangle \mathcal{B}_2\}$$

[As the conditional fuzzy expectation of a continuous function is uncoditional one.

$\{E[e^{u(\mathcal{B}(t+s) \ominus \mathcal{B}(t))}]| \vartriangle \mathcal{B}_1, \vartriangle \mathcal{B}_2\}$ is not depend on $\tilde{\mathcal{F}}_t$]

$$= \{e^{u\mathcal{B}(t)}| \vartriangle \mathcal{B}_1\} \otimes \{E[e^{u(\mathcal{B}(t+s) \ominus \mathcal{B}(t))}|\mathcal{B}(t)]| \vartriangle \mathcal{B}_1, \vartriangle \mathcal{B}_2\}$$

$$= \{e^{u\mathcal{B}(t)}| \vartriangle \mathcal{B}_1\} \otimes \{E[e^{u\mathcal{B}(t+s)}|\mathcal{B}(t)]| \vartriangle \mathcal{B}_2\} \otimes \{E[e^{-u\mathcal{B}(t)}|\mathcal{B}(t)]| \vartriangle \mathcal{B}_1\}$$

$$= \{e^{u\mathcal{B}(t)}| \vartriangle \mathcal{B}_1\} \otimes \{E[e^{u\mathcal{B}(t+s)}|\mathcal{B}(t)]| \vartriangle \mathcal{B}_2\} \otimes \{e^{-u\mathcal{B}(t)}| \vartriangle \mathcal{B}_1\}$$

$$= \{E[e^{u\mathcal{B}(t+s)}|\mathcal{B}(t)]| \vartriangle \mathcal{B}_2\}$$

Hence $\{E[e^{u\mathcal{B}(t+s)}|\tilde{\mathcal{F}}_t]| \vartriangle \mathcal{B}_2\}$=$\{E[e^{u\mathcal{B}(t+s)}|\mathcal{B}(t)]| \vartriangle \mathcal{B}_2\}$.

4 Fuzzy Martingale properties on FBM:-

Theorem 4.1

If $\tilde{\mathcal{B}}(t)$ is a FBM then

(i) $\tilde{\mathcal{B}}(t)$ is a FM.

(ii) $\tilde{\mathcal{B}}^2(t) \ominus \tilde{t}$ is a FM.

(iii) For any λ, $e^{\lambda\tilde{\mathcal{B}}(t) \ominus \frac{\lambda^2}{2}\tilde{t}}$ is a FM.

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Proof:- Let $\triangle \mathcal{B}_1 = \{\mathcal{B}(t)|\mathcal{B}(t) \in \widetilde{\mathcal{B}}_\alpha(t)\}$, $\triangle \mathcal{B}_2 = \{\mathcal{B}(s)|\mathcal{B}(s) \in \widetilde{\mathcal{B}}_\alpha(s)\}$

$$\triangle t = \{t|t \in \tilde{t}_\alpha\}, \triangle s = \{s|s \in \tilde{s}_\alpha\}$$

For each $\alpha \in [0,1]$ $\widetilde{\mathcal{B}}_\alpha(t)$ and $\widetilde{\mathcal{B}}_\alpha(s)$ are two interval valued BMs.

(i) The mean of FBM $\widetilde{\mathcal{B}}(t)$ is $\tilde{0}$ and variance is $\tilde{t}$.

So $\widetilde{\mathcal{B}}(t)$ be a fuzzy normal distribution i.e. $\widetilde{\mathcal{B}}(t) \sim \mathrm{N}(\tilde{0}, \tilde{t})$.

FBM $\widetilde{\mathcal{B}}(t)$ is integrable and $E[\widetilde{\mathcal{B}}(t)] = \tilde{0}$.

To prove $\widetilde{\mathcal{B}}(t)$ be FBM i.e for each $\alpha \in [0,1]$, $\{E[\mathcal{B}(t)|\tilde{\mathcal{F}}_s]| \triangle \mathcal{B}_1\} = \{\mathcal{B}(s)| \triangle \mathcal{B}_2\}$

The α-cut of FBM $\widetilde{\mathcal{B}}(t)$ can be written as

$\widetilde{\mathcal{B}}_\alpha(t) = \widetilde{\mathcal{B}}_\alpha(s) \oplus \widetilde{\mathcal{B}}_\alpha(t) \ominus \widetilde{\mathcal{B}}_\alpha(s)$.

So, $\{E[\mathcal{B}(t)|\tilde{\mathcal{F}}_s]| \triangle \mathcal{B}_1\} = \{E[(\mathcal{B}(s) \oplus \mathcal{B}(t) \ominus \mathcal{B}(s))|\tilde{\mathcal{F}}_s]| \triangle \mathcal{B}_1, \triangle \mathcal{B}_2\}$. (By F property)

$$= \{E[\mathcal{B}(s)|\tilde{\mathcal{F}}_s]| \triangle \mathcal{B}_2\} \oplus \{E[\mathcal{B}(t) \ominus \mathcal{B}(s)|\tilde{\mathcal{F}}_s]| \triangle \mathcal{B}_1, \triangle \mathcal{B}_2\}.$$

$$= \{\mathcal{B}(s)| \triangle \mathcal{B}_2\} \oplus \{E[\mathcal{B}(t) \ominus \mathcal{B}(s)]| \triangle \mathcal{B}_1, \triangle \mathcal{B}_2\}.$$

[As the conditional fuzzy expectation of a continuous function is uncoditional one i $\{E[(\mathcal{B}(t) \ominus \mathcal{B}(s))|\tilde{\mathcal{F}}_s]| \triangle \mathcal{B}_1, \triangle \mathcal{B}_2\}$ =$\{E[\mathcal{B}(t) \ominus \mathcal{B}(s)]| \triangle \mathcal{B}_1, \triangle \mathcal{B}_2\}$].

$$= \{\mathcal{B}(s)| \triangle \mathcal{B}_2\} \oplus \{E[\mathcal{B}(t)]| \triangle \mathcal{B}_1\} \ominus \{E[\mathcal{B}(s)]| \triangle \mathcal{B}_2\}$$

$$= \{\mathcal{B}(s)| \triangle \mathcal{B}_2\} \square$$

(ii) The variance of FBM is $\tilde{t}$ i.e $\{E[\mathcal{B}^2(t)]| \triangle \mathcal{B}_1\} = \{t| \triangle t\} \prec \infty$

So $\widetilde{\mathcal{B}}^2(t)$ is integrable.

To prove $\{E[\mathcal{B}(t)|\tilde{\mathcal{F}}_s]| \triangle \mathcal{B}_1\} = \{\mathcal{B}^2(s)| \triangle \mathcal{B}_1\} \ominus \{s| \triangle s\}$

The α-cut of FBM $\widetilde{\mathcal{B}}^2(t) \ominus t$ can be written as

$$\{\mathcal{B}^2(t) \ominus t| \triangle \mathcal{B}_1, \triangle t\} = \{\mathcal{B}^2(s) \oplus \mathcal{B}^2(t) \ominus \mathcal{B}^2(s) \ominus t| \triangle \mathcal{B}_1, \triangle \mathcal{B}_2, \triangle t\}.$$

$$= \{\mathcal{B}^2(s) \oplus [\mathcal{B}(t) \ominus \mathcal{B}(t)]^2 \oplus 2[\mathcal{B}(s) \otimes (\mathcal{B}(t) \ominus \mathcal{B}(s))] \ominus t| \triangle \mathcal{B}_1, \triangle \mathcal{B}_2, \triangle t\}.$$

$\{E[\mathcal{B}^2(t) \ominus t|\tilde{\mathcal{F}}_s]| \triangle \mathcal{B}_1, \triangle t\}$

$= \{E[(\mathcal{B}^2(s) \oplus [\mathcal{B}(t) \ominus \mathcal{B}(s)]^2 \oplus 2[\mathcal{B}(s) \otimes (\mathcal{B}(t) \ominus \mathcal{B}(s))] \in t)|\tilde{\mathcal{F}}_s]| \triangle \mathcal{B}_1, \triangle \mathcal{B}_2, \triangle t\}$ (By FM property)

$$= \{E(\mathcal{B}^2(s))|\tilde{\mathcal{F}}_s| \triangle \mathcal{B}_2\} \oplus \{E[\mathcal{B}(t) \ominus \mathcal{B}(s)]^2|\tilde{\mathcal{F}}_t| \triangle \mathcal{B}_1, \triangle \mathcal{B}_2\}$$
$$\oplus \{2E[\mathcal{B}(t) \otimes (\mathcal{B}(t) \ominus \mathcal{B}(s))|\tilde{\mathcal{F}}_s]| \triangle \mathcal{B}_1, \triangle \mathcal{B}_2\} \ominus \{t| \triangle t\}$$

$$= \{\mathcal{B}^2(s)| \triangle \mathcal{B}_2\} \oplus \{E[\mathcal{B}(t) \ominus \mathcal{B}(s)]^2| \triangle \mathcal{B}_1, \triangle \mathcal{B}_2\}$$
$$\oplus \{2(E[\mathcal{B}(s)|\tilde{\mathcal{F}}_s] \otimes E[\mathcal{B}(t) \ominus \mathcal{B}(s)|\tilde{\mathcal{F}}_s])| \triangle \mathcal{B}_1, \triangle \mathcal{B}_2\} \ominus \{t| \triangle t\}$$

$$= \{\mathcal{B}^2(s)| \triangle \mathcal{B}_1\} \oplus \{t \ominus s| \triangle t, \triangle s\} \oplus \{2(\mathcal{B}(s) \otimes (E[\mathcal{B}(t) \ominus \mathcal{B}(s)])| \triangle \mathcal{B}_1, \triangle \mathcal{B}_2\} \ominus \{t| \triangle t\}$$

$$= \{\mathcal{B}^2(s)| \triangle \mathcal{B}_2\} \oplus \{t| \triangle t\} \ominus \{s| \triangle s\} \ominus \{t| \triangle t\}$$

$$= \{\mathcal{B}^2(s)| \triangle \mathcal{B}_2\} \ominus \{s| \triangle s\} = \{\mathcal{B}^2(s) \ominus s| \triangle \mathcal{B}_2, \triangle s\}$$

So, $\{E[\mathcal{B}^2(t) \ominus t|\tilde{\mathcal{F}}_s]| \triangle \mathcal{B}_1, \triangle t\}$=$\{\mathcal{B}^2(s) \ominus s| \triangle \mathcal{B}_2, \triangle s\}$

Hence $\mathcal{B}^2(t) \ominus \tilde{t}$ is a FM. $\square$.

(iii) For each $\alpha \in [0,1]$, the moment generating function of FBM is

$$\{E[e^{\lambda\mathcal{B}(t)}]| \triangle \mathcal{B}_1\} = \{e^{\frac{\lambda^2}{2}t}| \triangle t\}$$

As the mean FBM is $\tilde{0}$ and variance is $\tilde{t}$,

So $\{e^{\lambda\widetilde{\mathcal{B}}(t) \ominus \frac{\lambda^2}{2}\tilde{t}}\}$ is integrable.

Now $\{e^{\lambda\mathcal{B}(t)}| \triangle \mathcal{B}_1\}$ =$\{e^{\lambda[\mathcal{B}(s) \oplus \mathcal{B}(t) \ominus \mathcal{B}(s)]}| \triangle \mathcal{B}_1, \triangle \mathcal{B}_2\}$

$$= \{e^{\lambda\mathcal{B}(s)}| \triangle \mathcal{B}_1\} \otimes \{e^{\lambda[\mathcal{B}(t) - \mathcal{B}(s)]}| \triangle \mathcal{B}_1, \triangle \mathcal{B}_2\}$$

Using FM, $\{E[e^{\lambda\mathcal{B}(t)}|\tilde{\mathcal{F}}_s]| \triangle \mathcal{B} \triangle \mathcal{B}_2\} = \{E[(e^{\lambda\mathcal{B}(s)} \otimes e^{\lambda(\mathcal{B}(t) \ominus \mathcal{B}(s))})|\tilde{\mathcal{F}}_s]| \triangle \mathcal{B}_1, \triangle \mathcal{B}_2\}$

$$= \{E[e^{\lambda\mathcal{B}(t)}|\tilde{\mathcal{F}}_s]| \triangle \mathcal{B}_1\} \otimes \{E[e^{\lambda(\mathcal{B}(t) \ominus \mathcal{B}(s))}|\tilde{\mathcal{F}}_s]| \triangle \mathcal{B}_1, \triangle \mathcal{B}_2\}$$

$$= \{e^{\lambda\mathcal{B}(s)}| \triangle \mathcal{B}_2\} \otimes \{E[e^{\lambda(\mathcal{B}(t) \ominus \mathcal{B}(s))}]| \triangle \mathcal{B}_1, \triangle \mathcal{B}_2\}$$

$$\{e^{\lambda\mathcal{B}(s)}| \triangle \mathcal{B}_2\} \otimes \{E[e^{\lambda\mathcal{B}(t)}]| \triangle \mathcal{B}_1\} \otimes \{E[e^{-\lambda\mathcal{B}(s)}]| \triangle \mathcal{B}_2\}$$

$= \{e^{\lambda\mathcal{B}(s)}| \triangle \mathcal{B}_2\} \otimes \{e^{\frac{\lambda^2}{2}(t \ominus s)}| \triangle t, \triangle s\}$ (Using MGF of FBM)

Now $\{E[(e^{\lambda\mathcal{B}(t) \ominus \frac{\lambda^2}{2}t})|\tilde{\mathcal{F}}_s]| \triangle \mathcal{B}_1, \triangle t\} = \{[(e^{\lambda\mathcal{B}(s)} \otimes e^{\frac{\lambda^2}{2}(t \ominus s)} \otimes e^{-\frac{\lambda^2}{2}t})]| \triangle \mathcal{B}_1, \triangle t\}$

$$= \{e^{\lambda\mathcal{B}(s) \ominus \frac{\lambda^2}{2}s}| \triangle \mathcal{B}_2, \triangle t\}$$

Hence For any λ, $e^{\lambda\widetilde{\mathcal{B}}(t) \ominus \frac{\lambda^2}{2}\tilde{t}}$ is a FM.

Theorem 4.2

Let $\widetilde{\mathcal{B}}(t)$ be a fuzzy continuous process for any λ, $\widetilde{\mathcal{B}}(t) \ominus e^{\frac{\lambda^2}{2}\tilde{t}}$ is a FM then $\widetilde{\mathcal{B}}(t)$ be a FBM.

Proof:- Let $\widetilde{\mathcal{B}}(t)$ be a fuzzy continuous process.

$\triangle \mathcal{B}_1 = \{\mathcal{B}(t)|\mathcal{B}(t) \in \widetilde{\mathcal{B}}_\alpha(t)\}$, $\triangle \mathcal{B}_2 = \{\mathcal{B}(s)|\mathcal{B}(s) \in \widetilde{\mathcal{B}}_\alpha(s)\}$

$$\triangle t = \{t|t \in (\tilde{t}_\alpha\}, \triangle s = \{s|s \in \tilde{s}_\alpha\}$$

For each $\alpha \in [0,1]$ $\widetilde{\mathcal{B}}_\alpha(t)$ and $\widetilde{\mathcal{B}}_\alpha(s)$ are interval valued BMs.

For any λ, $\{e^{\lambda\widetilde{\mathcal{B}}(t) \ominus \frac{\lambda^2}{2}\tilde{t}}\}$ is a FM. So $\{E[(e^{\lambda\mathcal{B}(t) \ominus \frac{\lambda^2}{2}t})|\tilde{\mathcal{F}}_s]| \triangle \mathcal{B}_1, \triangle t\} = \{e^{\lambda\mathcal{B}(s) \ominus \frac{\lambda^2}{2}s}| \triangle \mathcal{B}_2, \triangle s\}$ Now $\{E[e^{\lambda(\mathcal{B}(t) \ominus \mathcal{B}(s))}|\tilde{\mathcal{F}}_s]| \triangle \mathcal{B}_1, \triangle \mathcal{B}_2, \triangle t\}$

$$= \{E[e^{\lambda\mathcal{B}(t)}]| \triangle \mathcal{B}_1\} \ominus \{E[e^{\lambda\mathcal{B}(s)}]| \triangle \mathcal{B}_2\}$$

$$= \{e^{\frac{\lambda^2}{2}t}| \triangle t\} \ominus \{e^{\frac{\lambda^2}{2}s}| \triangle s\}$$

$$= \{e^{(t \ominus s)\frac{\lambda^2}{2}} \triangle t, \triangle s\}$$

The moment generating function of $\widetilde{\mathcal{B}}(t) \ominus \widetilde{\mathcal{B}}(s)$ is

$$\{E[e^{\lambda(\mathcal{B}(t) \ominus \mathcal{B}(s))}]| \triangle \mathcal{B}_1, \triangle \mathcal{B}_2\} = \{e^{(t \ominus s)\frac{\lambda^2}{2}}| \triangle t, \triangle s\}$$

Since $\widetilde{\mathcal{B}}(t) \ominus \widetilde{\mathcal{B}}(s) \sim \mathrm{N}\ (\tilde{0}, \tilde{t} \ominus \tilde{s})$ and $\widetilde{\mathcal{B}}(t) \ominus \widetilde{\mathcal{B}}(s)$ is independent of $\tilde{\mathcal{F}}_s$.

For $\tilde{u} \preccurlyeq \tilde{s}$, $\widetilde{\mathcal{B}}(u)$ is independent of $\tilde{\mathcal{F}}_s$.

Hence $\widetilde{\mathcal{B}}(t)$ is a FBM and $\widetilde{\mathcal{B}}_\alpha(t)$ is the α-cut of FBM, $\forall\alpha \in [0,1]$.

Theorem 4.3

Let $\widetilde{\mathcal{B}}(t)$ be a fuzzy continuous process with $\widetilde{\mathcal{B}}(0) = \tilde{0}$ and $\widetilde{\mathcal{B}}^2(t) \ominus \tilde{t}$ is a FM then $\widetilde{\mathcal{B}}(t)$ is a FBM.

Proof:- Let $\triangle \mathcal{B}_1 = \{\mathcal{B}(t)|\mathcal{B}(t) \in \widetilde{\mathcal{B}}_\alpha(t)\}$, $\triangle \mathcal{B}_2 = \{\mathcal{B}(s)|\mathcal{B}(s) \in \widetilde{\mathcal{B}}_\alpha(s)\}$

$\triangle t = \{t|t \in \tilde{t}_\alpha\}, \triangle s = \{s|s \in \tilde{s}_\alpha\}$ $\forall\alpha \in [0,1]$

$\widetilde{\mathcal{B}}(t)$ be a a fuzzy continuous process, for each $\alpha \in [0,1]$, $\widetilde{\mathcal{B}}_\alpha(0) = \tilde{0}$.

We have $\widetilde{\mathcal{B}}^2(t) \ominus \tilde{t}$ be a FM. So $\{E[(\mathcal{B}^2(t) \ominus t)|\tilde{\mathcal{F}}_s]| \triangle \mathcal{B}_1, \triangle t\} = \{\mathcal{B}^2(s) \ominus s| \triangle \mathcal{B}_2, \triangle s\}$ Let $\widetilde{M}(t) = \widetilde{\mathcal{B}}^2(t) \ominus t$ is a FM.

Now $\{E[\mathcal{B}^2(t) \ominus t]| \triangle \mathcal{B}_1, \triangle t\} = \{E[M(t)]| \triangle \mathcal{B}_1, \triangle t\}$

$$\{E[(\mathcal{B}(s) \ominus N(t)|)^2 \ominus t]^2| \triangle \mathcal{B}_2, \triangle t\}(\widetilde{N}(t) = \widetilde{\mathcal{B}}(t) \ominus \widetilde{\mathcal{B}}(s))$$
$$= \{\mathcal{B}^2(s) \oplus 2(\mathcal{B}(s) \otimes E[N(t)]) \oplus E[N^2(t)] \ominus t| \triangle \mathcal{B}_1, \triangle t\}$$
$$= \{\mathcal{B}^2(s) \oplus t \ominus s \ominus t| \triangle \mathcal{B}_2, \triangle t, \triangle s\}(E[\widetilde{N}(t)] = \tilde{0})$$
$$= \{\mathcal{B}^2(s) \ominus s| \triangle \mathcal{B}_2, \triangle s\} = \widetilde{M}(s).$$

So $\widetilde{\mathcal{B}}(t)$ is not depend on $\tilde{\mathcal{F}}_s$.

As $\widetilde{\mathcal{B}}(t)$ is not depend of $\tilde{\mathcal{F}}_s$, so $\widetilde{\mathcal{B}}(s)$ is not depend all $\tilde{\mathcal{F}}(s)$, $\forall\ s \preccurlyeq t$.

So $\widetilde{\mathcal{B}}(t)$ satisfies independent increment property. Hence $\widetilde{\mathcal{B}}(t)$ be FBM.

5 Conclusion:-

In this paper the properties of FBM have been discussed. Using α-cut technique the quadratic variation of FBM and FBM is a FGP with mean $\tilde{0}$ and covariance is $\min(\tilde{t}, \tilde{s})$ and converse is also true have been proved. Next the FBM is FMP have been discussed. Finally it is concluded that FBM is a FM.

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