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Fekete-Szegö inequality for a subclass involving the generalized $\varkappa$ -Mittag-Leffler functions

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Abstract

In fractional calculus, the Mittag-Leffler function plas a vital role and its not been superfluous up to now. Nowadays, application of Mittag-Leffler have been enlightening the theory of univalent functions. The aim of this paperis to derive the initial co-efficient estimation and the Fekete-Szegö inequality for the subclass of analytic function.

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I. INTRODUCTION

The classical Mittag-Leffler function is denoted by$E_\alpha(z)$ [7, 8] and is defined as

$$E_\alpha(z) = \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)} \quad \alpha \in \mathbb{C}, \quad \Re(\alpha) > 0.$$

The class of functions $f(z)$ which are analytic in the open unit disc$\mathbb{U}$ is denoted by $\mathcal{A}$ and is of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad z \in \mathbb{U} \quad (1.1)$$

which are normalized by $f(0) = 0$ and$f'(0) = 1$.The class of analytic and normalized univalent functions in $\mathbb{U}$ is denoted by$\mathcal{S}$.

For an two functions f and g which are analytic in $\mathbb{U}$, the function f is subordinate to g in $\mathbb{U}$ and is denoted by f(z) $\prec$g(z) if there exists a schwarz function ω, which are analytic in $\mathbb{U}$ with $\omega(0) = 0$ *and* $|\omega(z) < 1|$ such that f(z)=g($\omega(z)$), (z$\in\mathbb{U}$).

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0) \quad and \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\phi(z)$ be analytic, and let the Maclaurin series of $\phi(z)$ be given by

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots. \quad (1.2)$$

where all coefficients are real and $B_1 > 0$.

Lemma 1 *[9] $P(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part in $\mathbb{U}$ then for any complex number μ we have,*

$$|c_2 - \mu c_1^2| \le 2\max\{1, |1 - 2\mu|\}.$$

then $|p_k| \le 1, \ k \in N$ where P is the class of functions analytic in $\mathbb{U}$ for which $p(0) = 1$ and Re$(p(z)) > 0, (z \in \mathbb{U})$.

Recently Hameed Ur Rehman et al., [3] normalized the most generalized Mittag-Leffler function and defined the following,

$$L_{k,\sigma,\beta,\delta}^{\gamma,q} f(z) = z + \sum_{n=2}^{\infty} \frac{(\gamma)_n q, k\Gamma k(\sigma+\beta)\Gamma(\delta+1)}{(\gamma)_q, k\Gamma k(\sigma n+\beta)\Gamma(\delta+n)} a_n z^n. \qquad (1.3)$$

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where $\sigma, \beta, \gamma \in \mathcal{C}, \mathcal{R}(\sigma) > 0, \mathcal{R}(\beta) > 0, k \in \mathbb{R}$, γ is non-negative real number, nq is a positive integer $q \in (0,1) \cup \mathbb{N}$.

Definition 1 *A subclass of $\mathcal{A}$ consisting of functions is of the form (1.1), and satisfying the following condition can be denoted as $\mathcal{M}(\alpha, \phi)$*

$$\mathcal{M}(\alpha, \phi) = \left\{ f \in \mathcal{A}: \left[(1 - \alpha) \frac{z[L_{k,\sigma,\beta,\delta}^{\gamma,q} f(z)]'}{L_{k,\sigma,\beta,\delta}^{\gamma,q} f(z)} + \alpha \frac{[z(L_{k,\sigma,\beta,\delta}^{\gamma,q} f(z)')]'}{[L_{k,\sigma,\beta,\delta}^{\gamma,q} f(z)]'} \right] \prec \phi(z), \quad z \in \mathbb{U} \right\}$$

where $L_{k,\sigma,\beta,\delta}^{\gamma,q} f(z)$ is defined in (1.3).

Remark

By taking the suitable choices of the parameters, we get

$$L_{1,0,1,2}^{1,1} f(z) = \frac{2}{z} \int_0^z f(t) dt$$
$$= z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1} \right) a_n z^n$$
$$= -\frac{2 log(1-z)}{z} - 2.$$

which is the type of Bernardi Integral[1] and is the special case is studied by Libera [5] and Livingston[6].

II. INITIAL COEFFICIENTS

The first few coefficient estimates for the classes of $\mathcal{M}(\alpha, \phi)$ are derived in the following theorem.

Theorem 1 *If $f \in \mathcal{M}(\alpha, \phi)$ then,*

$$|a_2| \le \frac{B_1 c_1}{2(1+\alpha)A} \text{ and}$$

$$|a_3| \le \frac{1}{2(1+2\alpha)B} \left[\frac{1}{2} B_1 \left(c_2 - \frac{c_1^2}{2} + \frac{1}{4} B_2 c_1^2 + (1+2\alpha) \frac{B_1^2 c_1^2}{(1+\alpha)^2} \right) \right]$$

where

$$A = \frac{(\gamma)_2 q, k \Gamma k(\sigma+\beta) \Gamma(\delta+1)}{(\gamma)_q, k \Gamma k(\sigma n+\beta) \Gamma(\delta+2)}$$

and

$$B = \frac{(\gamma)_3 q, k \Gamma k(\sigma+\beta) \Gamma(\delta+1)}{(\gamma)_q, k \Gamma k(\sigma n+\beta) \Gamma(\delta+3)}$$

Proof. If $f \in \mathcal{M}(\alpha, \phi)$, then

$$(1-\alpha)[1 + A a_2 z + (2B a_3 - A^2 a_2^2) z^2 + (A^3 a_2^3 - 3AB a_2 a_3) z^3] + \alpha[1 + 2A a_2 z + (6B a_3 - 4A^2 a_2^2) z^2] = \phi(w(z) \quad (2.1)$$

If $p_1(z)$ is analytic and has positive real part in $\mathbb{U}$ and $p_1(0) = 1$, then define the functions $p_1(z)$ as

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots$$

From the above equation we obtain

$$w(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{c_1}{2} z + \frac{1}{2} \left(c_2 - \frac{c_1^2}{2} \right) z^2 + \cdots \quad (2.2)$$

Then p_1 is analytic in $\mathbb{U}$ with $p_1(0) - 1 = 0$ and has a positive real part in $\mathbb{U}$. By using (2.2) and (1.2), it is clear that

$$\phi\left(\frac{p_1(z)-1}{p_2(z)+1} \right) = 1 + \frac{B_1 c_1}{2} z + \left\{ \frac{B_1}{2} \left(c_2 - \frac{c_1^2}{2} \right) + \frac{B_2 c_1^2}{4} \right\} z^2 \ldots \quad (2.3)$$

Equating the co-efficients of like powers of z in (2.1), we obtain

$$a_2 = \frac{B_1 c_1}{2(1+\alpha)A} \quad (2.4)$$

$$a_3 = \frac{1}{2(1+2\alpha)B} \left[\frac{1}{2} B_1 \left(c_2 - \frac{c_1^2}{2} + \frac{1}{4} B_2 c_1^2 + (1+2\alpha) \frac{B_1^2 c_1^2}{(1+\alpha)^2} \right) \right] \quad (2.5)$$

III. THE FEKETE-SZEGÖ INEQUALITY

T **Theorem 2** *If $f \in M_\alpha(\alpha, \phi)$, then*

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$$|a_3 - \mu a_2^2| \le \frac{B_1}{2(1+2\alpha)B} \max\left\{1, \frac{B_2}{B_1} - \frac{B_1(1+2\alpha)}{(1+\alpha)^2}\left(2\mu\frac{B}{A^2} - 4B_1\right)\right\}$$

Proof. From (2.4) and (2.5) we get

$$a_3 - \mu a_2^2 = \frac{1}{2(1+2\alpha)B}\left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2} + \frac{1}{4}B_2c_1^2 + (1+2\alpha)\frac{B_1^2c_1^2}{(1+\alpha)^2}\right)\right] - \mu\frac{B_1^2c_1^2}{4(1+\alpha)^2A^2}.$$

By simple calculation we get

$$a_3 - \mu a_2^2 = \frac{B_1}{4(1+2\alpha)B}\left[c_2 - \frac{c_1^2}{2}\left[1 - \frac{B_2}{B_1} + \frac{B_1(1+2\alpha)}{(1+\alpha)^2}\left(2\mu\frac{B}{A^2} - 4B_1\right)\right]\right]$$

Hence, we have

$$a_3 - \mu a_2^2 = \frac{B_1}{4(1+2\alpha)B}[c_2 - \upsilon c_1^2] \qquad (3.1)$$

where

$$\upsilon = \frac{1}{2}\left[1 - \frac{B_2}{B_1} + \frac{B_1(1+2\alpha)}{(1+\alpha)^2}\left(2\mu\frac{B}{A^2} - 4B_1\right)\right]$$

By applying Lemma 1 to equation (3.1), we get the required result

Corollary 1 *When* $\alpha = 1$, *then*

$$|a_3 - \mu a_2^2| \le \frac{B_1}{12B}[c_2 - \upsilon c_1^2]$$

where,

$$\upsilon = \frac{1}{2}\left[1 - \frac{B_2}{B_1} + \frac{3B_1}{4}\left(2\mu\frac{B}{A^2} - 4B_1\right)\right].$$

Corollary 2 *When* $\alpha = 0$, *then*

$$|a_3 - \mu a_2^2| \le \frac{B_1}{4B}[c_2 - \upsilon c_1^2]$$

where,

$$\upsilon = \frac{1}{2}\left[1 - \frac{B_2}{B_1} + B_1\left(2\mu\frac{B}{A^2} - 4B_1\right)\right].$$

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