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The Boundary State Method in Solving the Anisotropic Elasticity Theory Problems for a Multi-Connected Flat Region

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Abstract:
The energy method of boundary states is used to solve plane problems of the theory of elasticity for an anisotropic body, which, in the general case, has non-circular cutouts. The body is in a flat state of tension.
The basis of the method is the concept of spaces of internal and boundary states, which are Hilbert and are conjugated by an isomorphism. The space of internal States includes components of the displacement vector, components of strain and stress tensors. The scalar product for the space of internal States expresses the internal energy of elastic deformation. The boundary state space includes displacements of the body boundary points and forces at the boundary. The scalar product in the space of boundary States expresses the action of surface forces. The General solution of the plane Lechnitzky problem for an anisotropic body is used to construct the basis of internal States, with usage of analytical functions for a multi-connected medium. Afterwards, the orthogonalization of the state spaces bases is performed with the decomposition of the desired elastic state into a Fourier series, where the basis States act as elements of the decomposition. Using the forces specified on the boundary, Fourier coefficients are derived and the solution is constructed.
The first basic problem for circular and rectangular plates with two circular notches is solved. A graphic illustration of the results is given and conclusions are drawn.

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1. INTRODUCTION

From the point of view of the theory of elasticity, modern materials used in aircraft manufacturing, construction, and mechanical engineering are characterized as anisotropic, that is, materials in which there is a difference in elastic properties for different directions. These include polymers, fiberglass reinforced polymers, polymers, polycrystalline metals, glass reinforcement, etc. Naturally, parts from such materials are somehow loaded and, to facilitate them, carry out technological cuts, which, of course, affect the general elastic state. Determining the stress-strain state of such materials is a difficult task, especially if these bodies have cutouts.

The main problems of the mechanics of anisotropic bodies are fully studied. At present, narrower issues are considered, for example, joint deformation by surface and mass forces [1, 2]. The peculiarity of the solution is that the resulting elastic simultaneously satisfies the conditions on the surface and inside the region. In [3], using the general representation of S. G. Lehnitsky, the problem of torsion of an extended anisotropic cylinder was solved. The solution was carried out using the boundary state method. The work [4] shows a method for writing out parametric solutions for weakly anisotropic plates. The solution was carried out using the boundary state method and the small parameter method.

The study of the deformation of anisotropic bodies has always been an actual direction in science. External forces can simulate not only uniaxial tension, but also torsion of solid and hollow anisotropic cylinders. In the working [5], a variant of flow theory was developed for the case of materials with large anisotropic elastoplastic deformations. The corresponding dynamic problem was formulated and a numerical method for solving two-dimensional axisymmetric problems was developed. A number of papers devoted to the torsion of bodies from anisotropic materials, for example, in the working [6], the stress-strain state of anisotropic cylindrical and prismatic rods was investigated under an arbitrary plasticity condition. In the working [7], a study of the stresses distribution peculiarities was undertaken and displacements in individual layers of a multilayer anisotropic rod. In the working [8], a method is proposed for solving the problem of layered anisotropic rods torsion by the finite element method. The problem of rods torsion of rhomboid section and section of compressor is considered. In the working [9], solutions analysis for torsion problems and stretching of nanotubes with two types of cylindrical anisotropy is given, which was theorized by S.G. Lehnitsky in the framework of the classical theory of elasticity.

2. FORMULATION OF THE PROBLEM

The equilibrium of an anisotropic plate with elastic symmetry planes parallel to the median is considered. Plastic surgery has, in general, non-circular cutouts. Forces are applied to the outer contour of the plate and to the contour of the notches (Fig. 1). The first main problem of the theory of elasticity is solved.

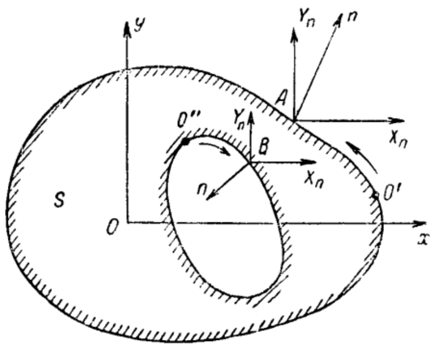

Figure 1: Anisotropic plate

The task is to determine the stress-strain state of the plate.

3. THE SOLUTION METHOD

To solve this problem, we use the boundary state method (BSM) [10]. BSM is a new energy method for solving problems of equations of mathematical physics. The method demonstrated its efficiency in solving boundary problems of the elasticity theory, both for isotropic and anisotropic media, in solving problems of thermoelasticity, hydrodynamics of an ideal fluid, dynamics (oscillations) of isotropic bodies.

The foundation of the method is the space of internal Ξ and boundary $\tilde{A}$ states:

$$\Xi = \{\xi_1, \xi_2, \xi_3, \ldots, \xi_k, \ldots\};\ \tilde{A} = \{\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_k, \ldots\}.$$

The internal state is determined by the sets of displacements vector components, tenors of deformations and stresses:

$$\xi_k = \{u_i^k, \varepsilon_{ij}^k, \sigma_{ij}^k\}. \quad (1)$$

The main difficulty in forming a solution in the BSM is the design of the basis for internal states, that relies on a common or fundamental solution for the environment; It is also possible to

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use any private or special solutions. The method of constructing the basis of internal states will be described below.

Scalar product in the space of internal states Ξ is expressed through the internal energy of elastic deformation (hence the membership of the method in the energy class). For example, for the first and second internal state of the body occupying the V region:

$$(\xi_1, \xi_2) = \int_V \varepsilon_{ij}^1 \sigma_{ij}^2 dV .$$

moreover, due to the commutatively of the states of the medium:

$$(\xi_1, \xi_2) = (\xi_2, \xi_1) = \int_V \varepsilon_{ij}^1 \sigma_{ij}^2 dV = \int_V \varepsilon_{ij}^2 \sigma_{ij}^1 dV .$$

The boundary state is determined by the components of the vector of displacement of the points of the boundary and surface forces:

$$\gamma_k = \{u_i^k, p_i^k\}, \; p_i^k = \sigma_{ij}^k n_j .$$

where n_j – is a component of the normal to the boundary.

In the space of boundary states Γ, the scalar product expresses the work of external forces on the surface of the body S, for example, for the first and second states:

$$(\gamma_1, \gamma_2) = \int_S p_i^1 u_i^2 dS .$$

moreover, by virtue of the principle of possible movements:

$$(\gamma_1, \gamma_2) = (\gamma_2, \gamma_1) = \int_S p_i^1 u_i^2 dS = \int_S p_i^2 u_i^1 dS .$$

It is proved that in the case of a smooth boundary both state spaces are Hilbert and are conjugated by an isomorphism [10]. By definition, each element of $\xi_k \in \Xi$ correspond to a single element of $\gamma_k \in \tilde{A}$, and this relationship exists on a one-to-one basis: $\xi_k \leftrightarrow \gamma_k$. This allows the internal state search to be reduced to the construction of a boundary state that is isomorphic to it. The latter essentially depends on the boundary conditions. In the case of the first and second main problems of mechanics, the problem recedes to resolving a system of equations for the Fourier coefficients, decomposition of the desired inner ξ and boundary γ states in a series in terms of the orthonormal basis elements:

$$\xi = \sum_{k=1}^{\infty} c_k \xi_k \; ; \; \gamma = \sum_{k=1}^{\infty} c_k \gamma_k .$$

or explicitly:

$$p_i = \sum_{k=1}^{\infty} c_k p_i^k \; ; \; u_i = \sum_{k=1}^{\infty} c_k u_i^k \; ; \; \sigma_{ij} = \sum_{k=1}^{\infty} c_k \sigma_{ij}^k \; ;$$

$$\varepsilon_{ij} = \sum_{k=1}^{\infty} c_k \varepsilon_{ij}^k .$$

The Fourier coefficients in the case of the first primary problem with the forces given by the ends of the cylinder $\mathbf{p} = \{p_{x0}, p_{y0}\}$ are:

$$c_k = (\mathbf{p}, \mathbf{u}^k) = \int_S (p_{x0} u^k + p_{y0} v^k) dS .$$

where p_{x0}, p_{y0} – set at the end of the effort, $\mathbf{u}^k = \{u^k, v^k, w^k\}$ – displacement vector in the basic element $\gamma_k = \{u_i^k, p_i^k\}$.

4. CONSTRUCTION OF THE BASIS OF INTERNAL STATES

The boundary state method is based on a fundamental or general solution for the medium. In the case of plane problems of anisotropic elasticity, the general solution is the complex representation formulas constructed by Lekhnitsky [11], which in the case of different roots of the characteristic equation:

$$\begin{aligned} \sigma_x &= 2\,\mathrm{Re}[\mu_1^2 \Phi_1'(z_1) + \mu_2^2 \Phi_2'(z_2)]; \\ \sigma_y &= 2\,\mathrm{Re}[\Phi_1'(z_1) + \Phi_2'(z_2)]; \\ \tau_{xy} &= -2\,\mathrm{Re}[\mu_1 \Phi_1'(z_1) + \mu_2 \Phi_2'(z_2)]; \\ u &= 2\,\mathrm{Re}[p_1 \Phi_1(z_1) + p_2 \Phi_2(z_2)]; \\ v &= 2\,\mathrm{Re}[q_1 \Phi_1(z_1) + q_2 \Phi_2(z_2)]. \end{aligned} \tag{2}$$

$$p_1 = a_{11}\mu_1^2 + a_{12} - a_{16}\mu_1 \; ; \; p_2 = a_{11}\mu_2^2 + a_{12} - a_{16}\mu_2 \; ;$$

$$q_1 = a_{12}\mu_1 + a_{22}/\mu_1 - a_{26} \; ; \; q_2 = a_{12}\mu_2 + a_{22}/\mu_2 - a_{26} ,$$

where a_{ij} – are the deformation constants; $z_1 = x_1 + \mu_1 y$, $z_2 = x_2 + \mu_2 y$ – generalized complex variables; μ_1, μ_2 – are the complex roots of the characteristic equation [12–13] defined by the

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anisotropy parameters (if $\mu_1 = \mu_2$ and both roots are purely imaginary, then the medium is orthotropic with the planes of elastic symmetry normal to the one under consideration; if $\mu_1 = \mu_2 = i$, the medium is isotropic).

Complex potentials:

$$\Phi_1(z_1) = \frac{dF_1}{dz_1};\ \Phi_2(z_2) = \frac{dF_2}{dz_2};$$

$$\Phi_1'(z_1) = \frac{d\Phi_1}{dz_1};\ \Phi_2'(z_2) = \frac{d\Phi_2}{dz_2};$$

where $F = F_1 + F_2$ – stress function.

Functions $\Phi_1(z_1)$, $\Phi_2(z_2)$ are defined over regions D_1 and D_2, respectively, in complex planes z_1, z_2. If in the physical plane Oxy the body occupies the region D_0, then D_0 and D_2 are obtained from D_0 by the corresponding affine transformations; integration constants u_0, v_0 are responsible for the displacement and rotation of the body as a whole [6].

Let the main vector of the applied forces $J_i = \{P_{xk}, P_{yk}\}$ be determined on the inner contour of the multiply connected region (Fig. 2).

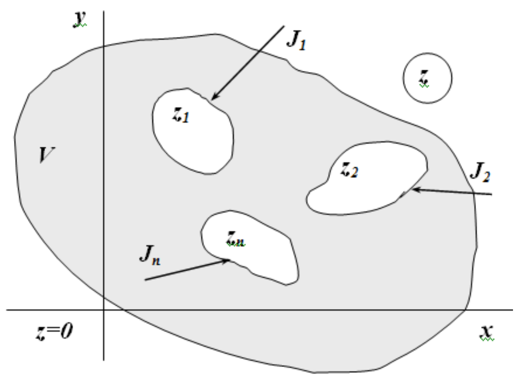


Figure 2: Multiply connected flat area

The representation for the Lechnitzcoco functions is as follows [11]:

$$\Phi_1(z_1) = \sum_{k=1}^{N} M_{1k} \ln(z_1 - z_{1n}) + \varphi(z_1);$$
$$\Phi_2(z_2) = \sum_{k=1}^{N} M_{2k} \ln(z_2 - z_{2n}) + \psi(z_2), \quad (3)$$

where M_{1k} and M_{2k} are complex constants determined from the system of equations:

$$M_{1k} - \overline{M}_{1k} + M_{2k} - \overline{M}_{2k} = \frac{1}{2\pi i} P_{yk};$$
$$\mu_1 M_{1k} - \bar{\mu}_1 \overline{M}_{1k} + \mu_2 M_{2k} - \bar{\mu}_2 \overline{M}_{2k} = -\frac{1}{2\pi i} P_{xk};$$
$$\mu_1^2 M_{1k} - \bar{\mu}_1^2 \overline{M}_{1k} + \mu_2^2 M_{2k} - \bar{\mu}_2^2 \overline{M}_{2k} = -\frac{1}{2\pi i a_{11}}\left(a_{16} P_{xk} + a_{12} P_{yk}\right);$$
$$\frac{1}{\mu_1} M_{1k} - \frac{1}{\bar{\mu}_1} \overline{M}_{1k} + \frac{1}{\mu_2} M_{2k} - \frac{1}{\bar{\mu}_2} \overline{M}_{2k} = -\frac{1}{2\pi i a_{22}}\left(a_{12} P_{xk} + a_{26} P_{yk}\right).$$

where P_{xk} and P_{yk} are the components of the main force acting on the inner loop of the k-th hole.

Basic sets of internal states can be constructed by generating possible options for analytic functions. For a multiply connected region, it has the form:

$$\begin{pmatrix}\varphi(z_1)\\ \psi(z_2)\end{pmatrix} \in$$
$$\left\{\begin{pmatrix}z_1\\ 0\end{pmatrix}, \begin{pmatrix}0\\ z_2\end{pmatrix}, \begin{pmatrix}iz_1\\ 0\end{pmatrix}, \begin{pmatrix}0\\ iz_2\end{pmatrix}, \begin{pmatrix}(z_1 - z_{1k})^{-1}\\ 0\end{pmatrix}, \begin{pmatrix}0\\ (z_2 - z_{2k})^{-1}\end{pmatrix}, \begin{pmatrix}i(z_1 - z_{1k})^{-1}\\ 0\end{pmatrix}, \begin{pmatrix}0\\ i(z_2 - z_{2k})^{-1}\end{pmatrix}\right\},$$
$$k = 1,2,..N. \quad (4)$$

The expressions for the analytic functions (4) are substituted in (3) and then in (2), thereby determining all the components of the plane stress state and forming the basis of internal states (1). Next comes the orthogonalization of the bases of the state spaces.

5. THE SOLUTION OF THE PROBLEM

The solution of the first main problem was carried out for a circular body in terms of a body with two circular cutouts (Fig. 3) and for a rectangular body with two circular cutouts (Fig. 5). It is assumed that the plate is made of laminated fiberglass orthogonal reinforcement with the following elastic characteristics (dimensionless form of solution adopted): Young's modules $E_X = 3{,}68$; $E_y = 2{,}68$; shear modulus $G = 0{,}5$; Poisson's ratio $\mu = 0{,}1$. The anisotropy axes relative to the *x* and *y* coordinate axes are rotated at an angle of $\pi/6$ counterclockwise.

On the external contour of a circular body, unit forces were imitated that simulated comprehensive compression; on internal contours comprehensive compression.

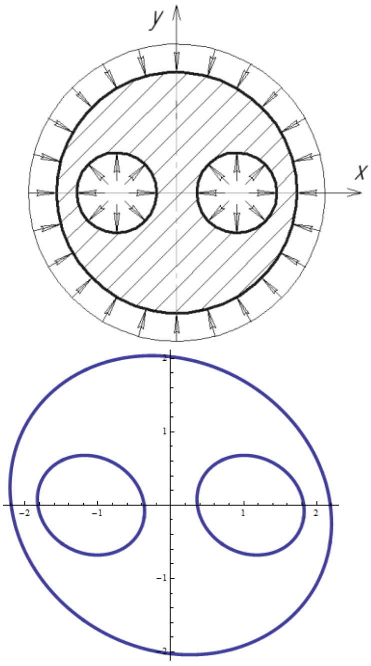
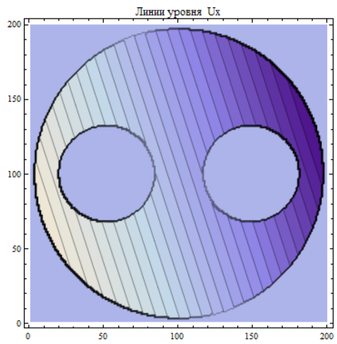
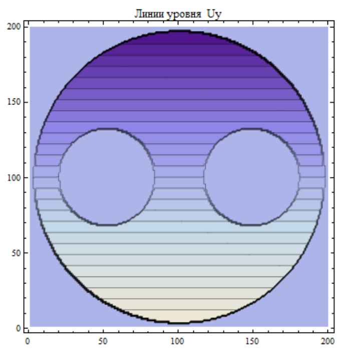


Figure 3: Boundary conditions for a circular body (left) and the contour of the deformed state (right)

Figure 4 shows the contour components of the displacement vector.



a



b

Figure 4: Isolines: *a* – displacement vector components u , *b* – displacement vector components v

For a rectangular-shaped body with cut-outs, forces imitating bending were set.

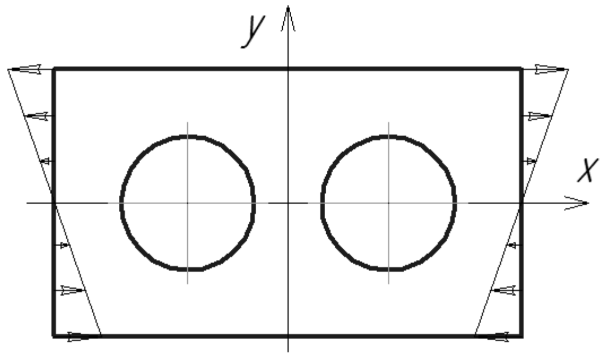
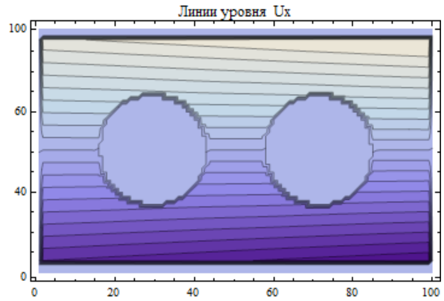
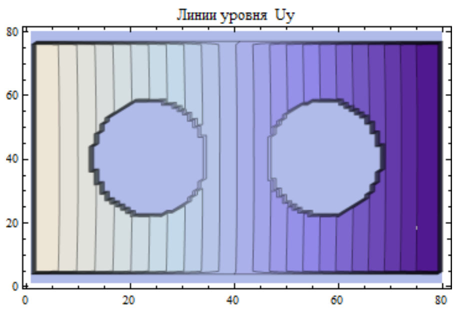


Figure 5: Boundary conditions for a rectangular body

Figure 6 shows the isolines of the components of the displacement vector of the stress tensor.



a



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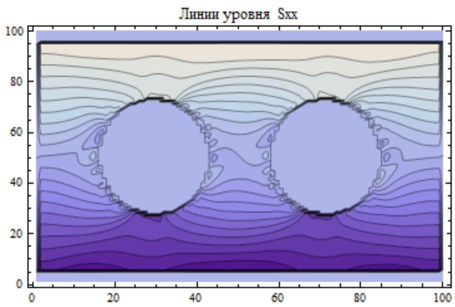

v

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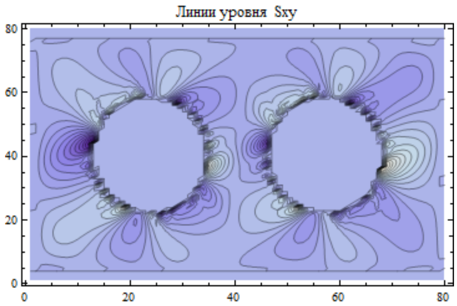

g

Figure 6: Isolines: *a* – displacement vector components *u*, *b* – displacement vector components *v*, *v* – stress tensor component σ_{xx} , *g* – stress tensor component σ_{xy} .

Thus, the boundary state method has been successfully implemented in terms of solving the first main problem for a multiply connected plane domain; the solution boils down to routine quadrature computation. A methodology has been developed for constructing the bases of state spaces for a plane multiply connected region, based on the existence of a common solution for an anisotropic medium. Concrete solutions to flat problems are constructed. When solving these problems, a sufficiently "long" (up to 300 elements) basis is required. The peculiarity of the solution is that the orthonormal basis for the body is constructed once and can be used to solve various boundary value problems.

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