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Portfolio Optimization under Multi-Period Scenario in the Uncertain Environment using Neural Network

Sunil Kumar Mittal[1,*], Namita Srivastava[2]
[1,2] Department of Mathematics, Maulana Azad National Institute of Technology, Bhopal, India
sunilmittal3639@gmail.com[1]
sri.namita@gmai.com[2]
Corresponding author*

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Abstract:
In this paper, a portfolio optimization problem is proposed in an uncertain environment. For this portfolio optimization problem, stocks are assumed to function as zigzag uncertain variables. Transaction price is also included in the optimized version. A mean-VaR (value at risk) bi-objective portfolio optimization model is devised to account for market uncertainty. Cardinality, bounding restrictions, and liquidity are considered in addition to risk and return to make the model more effective. A gradient-based neural network approach is applied to solve the planned model. Finally, an example portfolio is presented to display the efficacy and the feasibility of the model suggested in this paper.

Keywords: *Zigzag Uncertain variable, Multi-period portfolio, Uncertain multi-objective programming, Value at risk, Neural Networks*

I. INTRODUCTION

Portfolio optimization problems deal with individuals who want to invest their capital in the stock market to achieve personal investment goals. In financial planning's infancy, observation and experience were the two main pillars for one's success in this field. However, scientific developments in the field have provided us with tools to make better predictions, even for those of us who lack experience.

Modern portfolio theory's fundamental basis was set by Markowitz, [7] who developed the mean-variance model for portfolio optimization. The essence of the Markowitz model is the minimization of risk and the maximization of returns. Since this model was first put forth, many developments have been applied within a horizon of one period. In the real world, however, portfolio optimization approaches are generally multi-period approaches, as investors, in most cases, prefer to revise their capital allocation occasionally. Therefore, it is necessary to change the approach from a single-period portfolio to a multi-period portfolio.

Mossin [13], Merton, [27] and Samuelson [24] have already considered multi-period portfolio selection problems. Meanwhile, Li and Ng [5] have formulated a multi-period portfolio optimization method by taking mean variance as the primary parameter. Zhu et al. [28] proposed a dynamic and highly generalized version of the mean-variance model by introducing a bankruptcy constraint. Gulpinar and Rustem [19] developed a worst-case design for multiple periods for a stochastic market. Furthermore, a novel mean-semivariance-CVaR model for multi-period portfolios was proposed by Najafi and Mushakhian [1]; this model was designed to handle uncertainty in the stock market. Najafi and Pourahmadi [2] applied an efficient heuristics method to maximize the final utility of the investor using a multi-period portfolio in an uncertain environment. Hassanlou [14] introduced the concept of the chance constraint to the existing multi-period portfolio models. In his work, borrowing and lending rates were assumed to be different from one another and to be normally distributed random variables. Additionally, he applied a genetic algorithm to solve the model.

The majority of portfolio optimization models that have been constructed previously have used probability theory to determine the risk and returns associated with stocks. However, these models may not work in the existing financial market, as the market contains many non-probabilistic aspects that affect the fluctuation of stocks. Moreover, probability theory is not entirely suitable to handle the uncertainty of the financial market.

To resolve this issue, the fuzzy set theory was proposed in 1965 by Zadeh [15] and proved to be useful in managing variations in the financial market. Wang [30] developed a fuzzy-theory-based portfolio optimization model, while Huang [12] and Qin et al. [26] investigated credibility measures in a fuzzy portfolio selection model.

Although the fuzzy theory worked well in the field of portfolio optimization, an inconstancy in this theory has been pointed out by Liu [16], who explained that fuzzy theory fails to describe the subjectivity involved in the financial market. He proposed a new theory called uncertainty theory. Several researchers have examined uncertainty theory within the portfolio optimization problem. A mean-variance model for portfolio optimization in an uncertain environment was established by Qin et al. [25]. To further develop the model, Huang [11] introduced a risk curve in the mean risk model. Huang [10] later extended his research by incorporating a risk index into his existing work. Liu and Qin [18] then introduced a downside risk measure in the uncertain portfolio optimization problem.

Artificial intelligence techniques, such as the artificial neural network [5, 6, 9, 17, 20, 21, 22, 23, 29, 31], are powerful tools used to deal with various types of optimization problems. Artificial recurrent neural networks can transform an optimization problem into a dynamic system called the KKT (Karush Kuhn Tucker) system. The KKT system consists of first-order differential equations. Furthermore, the KKT system [3] can be transformed into an unconstrained optimization problem that approaches its optimal state at the same points at which the solution of the original problem lies.

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In this paper, five criteria (return, risk, liquidity, cardinality, and threshold) are used to investigate the portfolio optimization problem in a multi-period scenario. Uncertain mean values are used to evaluate returns, and risks are calculated as the value-at-risk of uncertain returns. Liquidity is measuredin terms of the turnover rates of stocks. Transaction cost is also considered in the proposed model. An uncertain multi-objective approach is adapted to remodel a multi-objective portfolio problem into a single-objective one. Finally, a gradient-based neural network strategy is employed to achieve the best results possible.

The structure of the paper is as follows. In Part 2, some basics of uncertainty theory are discussed. The design of a portfolio optimization model in a multi-period scenario with transaction costs is explained in the next section. In Part 3, the multi-objective portfolio problem is transformed into a single-objective problem by using uncertain multi-objective programming. Part 5 includes the use of the gradient descent algorithm of the neural network to obtain optimal results. For illustrative purposes, an experiment is performed on the stock prices taken from BSE India's website; the results are analyzed under different parameters in Part 6. Concluding remarks are given in the final section.

II. PRELIMINARIES

Some elementary concepts of uncertainty theory [15]are discussed in this section, which will be required in further segments.

Let us consider a measurable space $(\Gamma, \mathscr{L})$, Here Γbe a non-empty set and $\mathscr{L}$be a σ algebra defined on it. In uncertainty theory, each element Λ of $\mathscr{L}$ is defined as an event. To calculate the belief degree (chance of occurrence) of Λ, a number $\mathrm{M}\{\Lambda\}$ is assigned to each Λ in such a way that itsatisfiesfollowingAxioms,

Axiom 1.$\mathscr{M}\{\Gamma\} = 1$*(Normality)*

Axiom 2. If $\Lambda_1 \subset \Lambda_2$ then$\mathscr{M}\{\Lambda_1\} \leq \mathscr{M}\{\Lambda_2\}$*(Monotonicity)*

Axiom 3.$\mathscr{M}\{\Lambda\} + \mathscr{M}\{\Lambda^c\} = 1$*(Self-Duality)*

Axiom 4.If$\{\Lambda_i\}$ is the countable sequence of events then:

$$\mathscr{M}\{\cup_{i=1}^{\infty} \Lambda_i\} \leq \sum_{i=1}^{\infty} \mathscr{M}\{\Lambda_i\}. \quad \textit{(Countable Subadditivity)} \tag{1}$$

Remark 2.1 Here $\mathscr{M}$ is defined asan uncertain measure that is interpreted as belief degree of an uncertain event that may or may not happen.

Definition 2.2(Liu, [15]) If $\Gamma$is a non-empty set, $\mathscr{L}$ be a σ algebra over it and $\mathscr{M}$ be an uncertain measure. Then the triplet $(\Gamma, \mathscr{L}, \mathscr{M})$ is called an uncertainty space.

Definition 2.3(Liu, [15]) An uncertain variable ξ is a measurable function from an uncertainty space to the set of real numbers, such that $\{\xi \epsilon B\}$ is an event for any Boral set B of real numbers. The uncertain variable ξ has a uncertain distribution

$\Delta(u) = \mathscr{M}\{\xi \leq u\}$, $\Delta: \mathbb{R} \to [0,1]$

For Example,uncertainty distribution for

- linear uncertain variable is given by

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$$\Delta(s) = \begin{cases} 0 & if\ s \leq \ell, \\ \dfrac{(s-\ell)}{(u-\ell)} & if\ \ell \leq s \leq u, \\ 1 & if\ s \geq u. \end{cases}$$

$\mathscr{L}(a,b)$= notation used for Linear uncertain variable ($a < b, a$ and b are real numbers)

- Zigzag uncertain variable is given by

$$\Delta(s) = \begin{cases} 0 & if\ s \leq \ell, \\ \dfrac{(s-\ell)}{2(u-\ell)} & if\ \ell \leq s \leq u, \\ \dfrac{\ell+v-2u}{2(v-\ell)} & if\ u \leq s \leq v \\ 1 & if\ s \geq v \end{cases} \quad (2)$$

$\mathcal{Z}(a,b,c)$=notation used for Zigzag uncertain variable ($a < b < c, a, b\ and\ c$ are the real numbers)

- Normal uncertain variable is given by

$$\Delta(s) = \left(1 + \exp\left(\frac{\pi(e-s)}{\sqrt{3}\sigma}\right)\right)^{-1}, s\epsilon\mathbb{R}.$$

$\mathcal{N}(e,\sigma)$=notation used for normal uncertain variable ($\sigma > 0$,e and σ are the real numbers)

$\xi_1, \xi_2 \dots \xi_m$ are the independent uncertain variable if

$$\mathscr{M}\left\{\bigcap_{i=1}^{k}\{\xi_i \epsilon B_i\}\right\} = \min_{1\leq i\leq k} \mathscr{M}\{\xi_i \epsilon B_i\} \quad (3)$$

For any Borel sets $B_i\ (i = 1,2 \dots k)$ of real numbers.

Definition 2.4(Liu, [15]) Expected value of an uncertain variable ξ is given by

$$E[\xi] = \int_0^{\infty} \mathscr{M}\{\xi \geq s\}ds - \int_{-\infty}^{0} \mathscr{M}\{\xi \leq s\}ds \quad (4)$$

provided that at least one of the two integralsis finite.

In regard to Eq. (4), the expected value of a linear uncertain variable $\xi \sim \mathscr{L}(\alpha,\beta)is\frac{(\ell+u)}{2}$;the expected value of azigzag uncertain variable$\xi \sim \mathcal{Z}(\ell,u,v)$is $\frac{(\ell+2u+v)}{4}$; theexpected value of normal uncertain variable $\xi \sim \mathcal{N}(e,\sigma)ise$.

Theorem 2.5. (Liu, [15])Let we have two independent uncertain variables ξ and $\eta$with finite expected values.Then,

$$E[\alpha\xi + \beta\eta] = \alpha E[\xi] + \beta E[\eta] \quad (5)$$

for any real numbers α and β.

Theorem 2.6. (Liu, [15])Let an uncertain variable ξhas finite expected value. Then,

$$E[\alpha\xi + \beta] = \alpha E[\xi] + \beta \quad (6)$$

for any real numbers α and β.

Definition 2.7 Value at risk (VaR) of an uncertain variable $\xi$is the function $Var_\xi(\alpha)\colon (0,1] \to \mathbb{R}$ such that

$$VaR_\xi(\alpha) = \sup\{x|\mathscr{M}\{\xi \geq x\} \geq \alpha\}$$

Here $\alpha\epsilon(0,1]$ is called risk confidence level.

It can be rewritten as

$$VaR_\xi(\alpha) = \sup\{x|\mathscr{M}\{\xi \leq x\} \leq 1-\alpha\} \quad (7)$$

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Since the definition of the uncertainty distribution

$\Delta(u) = \mathscr{M}\{\xi \leq u\}$.

So from equation (6), we have

$$VaR_{\xi}(\alpha) = \sup\{x|\Delta(x) \leq 1 - \alpha\}$$

It can be rewritten in the form of inverse uncertainty distribution $\Delta^{-1}(\alpha)$as

$$VaR_{\xi}(\alpha) = \Delta^{-1}(1 - \alpha) \qquad (8)$$

Clearly$\mathrm{VaR}_{\xi}(\alpha)$ is a monotonically decreasing function ofα

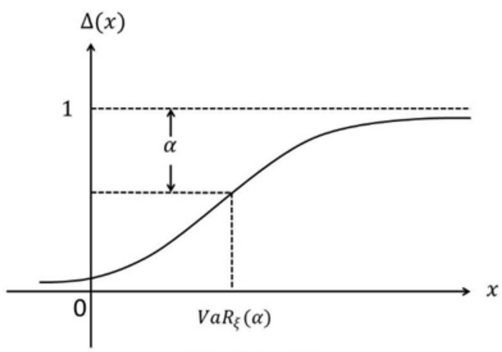

Fig.1. Value at risk

Let $\xi \sim \mathcal{Z}(a, b, c)$ be a zigzag uncertain variable, whose uncertainty distribution is given by Eq. (2). Now for any distinctive confidence level α with $0 < \alpha \leq 1$, the expression for uncertain value at risk function can be defined as

$$VaR_{\xi}(\alpha) = \begin{cases} 2(a-b)\alpha + 2b - a & if\ \alpha < 0.5, \\ 2(b-c)\alpha + c & if\ \alpha \geq 0.5. \end{cases} \qquad (9)$$

It should be noted that for less than 50% risk confidence level, $\alpha \geq 0.5$ will be used and for more than 50% risk confidence level, $\alpha < 0.5$ will be used.

III. MULTI-PERIOD PORTFOLIO OPTIMIZATION MODEL FORMULATION IN UNCERTAIN ENVIRONMENT

In this section, a multi-period portfolio optimization model in an uncertain environment is constructed. First, a brief description of the problem is given along with the notations that will be used in subsequent sections. Then, the uncertain returns and value-at-risk values for the given portfolio are calculated. Finally, some constraints are introduced so that the model complies with real-world situations.

3.1. Problem description and notations

The portfolio optimization problem is regarded as a dynamic affair in that trades take place in discrete time periods. In this paper, a portfolio which contains a combination of risky and risk-free assets is analyzed. In the proposed model, the initial wealth of the investor is W_1, and the investor can readjust this value at the beginning of the period T_1. Furthermore, the returns of risky assets are designed to function as a zigzag uncertain variable. The notations for different terminologies are given below.

x_{it} =Fraction of capital assigned to i^{th} stock at t time period;

x_{ft} = Fraction of capital assigned to risk-free asset at t time period;

r_{it}= i^{th}risky asset's return at time period t;

r_{ft} =Risk free asset's return at time period t;

r_{pt} =Portfolio's total return at time period t;

r_{Nt} =Portfolio's net return at time period t;

W_t=Wealth accumulated after t time period;

c_{it} =The unit tradingprice of risky i^{th}asset i at t time period;

L_{it}=The uncertain turnover rate of risky asset i at period t;

β= confidence level for annual turnover rates (Liquidity)

z_{it} = ainteger (binary) variable, where

$$z_{it} = \begin{cases} 1, & if\ risky\ asset\ i\ is\ contained\ in\ the\ portfolio \\ 0 & otherwise \end{cases}$$

$K =$ The number of assets that an investor wishes to include in the portfolio;

l_{it} =Minimum fraction of wealth that could be assigned to a stock.

u_{it} = Maximum fraction of wealth that could be assigned to a stock.

3.2. Objective functions

3.2.1 Maximization of final Wealth

The expected return of the portfolio in multi-period scenario is given by

$$E(r_{pt}) = \sum_{i=1}^{n} E(r_{it})x_{it} + r_{ft}x_{ft} \qquad (10)$$

Trading expensesis calculated for per unit time difference. The trading expenses fori^{th} risky asset at t time periodis $c_{it}|x_{it} - x_{i(t-1)}|$. Hence total trading cost of the portfolio is given by

$$C = \sum_{i=1}^{n} c_{it}|x_{it} - x_{i(t-1)}| \qquad (11)$$

Hence portfolio's net return at t time period is given by

$$E(r_{Nt}) = \sum_{i=1}^{n} E(r_{it})x_{it} + r_{ft}x_{ft} - \sum_{i=1}^{n} c_{it}|x_{it} - x_{i(t-1)}| \qquad (12)$$

Now from equation (12), the expected wealth at the beginning of period $t + 1$ is determined by

$$W_{t+1} = W_t(1 + E(r_{Nt}))$$

$$= W_t\left(1 + \left(\sum_{i=1}^{n} E(r_{it})x_{it} + r_{ft}x_{ft} - \sum_{i=1}^{n} c_{it}|x_{it} - x_{i(t-1)}|\right)\right)$$

$$= W_1 \prod_{t=1}^{T}\left(1 + \left(\sum_{i=1}^{n} E(r_{it})x_{it} + r_{ft}x_{ft} - \sum_{i=1}^{n} c_{it}|x_{it} - x_{i(t-1)}|\right)\right) \qquad (13)$$

3.2.2 Minimization of Value-at-Risk

Objective function for minimization of Value-at-risk is as follows

$$Min\ VaR_{\xi}(\alpha) \qquad (14)$$

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3.3constraints

3.3.1 Liquidity

Liquidity constraint is given by

$$\mathcal{M}\left(\sum_{i=1}^{n} L_{it}x_{it} \geq L\right) \geq \beta \qquad (0.5 < \beta \leq 1) \tag{15}$$

Here L is the minimum liquidity that an investor desires in any stocks.

3.3.2 Cardinality

This constraint allows us to diversify our investment and avoid capital flow in one or two higher return stocks. Total number of stocks on which an investor is allowed to distribute their capital is determined by

$$\sum_{i=1}^{n} z_{it} = K \tag{16}$$

3.3.3 Bounding constraint

The lower and upper bounds for i^{th} risky stock is determined by

$$l_{it}z_{it} \leq x_{it} \leq u_{it}z_{it} \tag{17}$$

Thus, a portfolio optimization model in multi-period scenario under uncertain environment will be as follows

P(1) $\quad Max\ W_{t+1} Min\ VaR_{\xi}(\alpha)$

Subject to

$$\mathcal{M}\left(\sum_{i=1}^{n} L_{it}x_{it} \geq L\right) \geq \beta$$

$$\sum_{i=1}^{n} z_{it} = K$$

$$l_{it}z_{it} \leq x_{it} \leq u_{it}z_{it}$$

$$z_{it} = \begin{cases} 1, & if\ asset\ i\ is\ contained\ in\ the\ portfolio \\ 0, & otherwise \end{cases}$$

$$i = 1,2,\ldots,n \quad t = 1,2,\ldots,T$$

3.4 Crisp Equivalents

In this paper, the stock returns and Liquidity (Turnover rate) are considered as zigzag uncertain variables with the triplet $r_{it} = (a_{it}, b_{it}, c_{it})$ and $L_{it} = (L_{ait}, L_{bit}, L_{cit})$ respectively. Now from equation (9) and (13), P(1) will be rewritten as follows

$$Max\ W_1 \prod_{t=1}^{T}\left(1 + \left(\sum_{i=1}^{n} E(r_{it})x_{it} + r_{ft}x_{ft} - \sum_{i=1}^{n} c_{it}\left|x_{it} - x_{i(t-1)}\right|\right)\right)$$

$$Min \sum_{i=1}^{n}[2(a_{it} - b_{it})\alpha + 2b_{it} - a_{it}]\, x_{it}$$

Zhu and Zhang[30] provided a method by which chance constraint of portfolio turnover rate can be transformed into equivalent another expression that is as follows

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$$\mathcal{M}\{\xi \geq r\} \geq \lambda \Leftrightarrow r \leq (2\lambda - 1)a + 2(1-\lambda)b \qquad (18)$$

Where, $\xi = \mathcal{Z}(a,b,c)\ with\ a < b < c$ is a zigzag uncertain variable and $\lambda(0.5 < \lambda \leq 1)$ is given uncertain confidence level.

From (18), the chance constraint of portfolio turnover rate can be replaced by equivalent constraint as follows

$$\mathcal{M}\left(\sum_{i=1}^{n} L_{it}x_{it} \geq L\right) \geq \beta \Rightarrow$$

$$\sum_{i=1}^{n}\big((2\beta-1)L_{ait} + (2-2\beta)L_{bit}\big)x_{it} \geq L$$

Now crisp equivalence of P(1) is defined as follows

P(2)

$$Max\ W_1 \prod_{t=1}^{T}\left(1 + \left(\sum_{i=1}^{n} E(r_{it})x_{it} + r_{ft}x_{ft} - \sum_{i=1}^{n} c_{it}\left|x_{it} - x_{i(t-1)}\right|\right)\right) \qquad (19)$$

$$Min \sum_{i=1}^{n}[2(a_{it} - b_{it})\alpha + 2b_{it} - a_{it}]\,x_{it} \qquad (20)$$

Subject to

$$\sum_{i=1}^{n}\big((2\beta-1)L_{ait} + (2-2\beta)L_{bit}\big)x_{it} \geq L \qquad (21)$$

$$\sum_{i=1}^{n} z_{it} = K \qquad (22)$$

$$l_{it}z_{it} \leq x_{it} \leq u_{it}z_{it} \qquad (23)$$

$$z_{it} = \begin{cases} 1, & if\ stocks\ included\ in\ portfoli \\ 0, & otherwise \end{cases} \qquad (24)$$

$$i = 1,2,\dots,n \quad t = 1,2,\dots,T \qquad (25)$$

IV. UNCERTAIN MULTI-OBJECTIVE PROGRAMMING

Liu [15] described a compromise model through which a multi-objective uncertain programming problem can be transformed into a single-objective programming problem. The optimality condition for the compromised solution is Pareto to the original problem.

The solution methodology for a compromise model of P(2) consist of four steps that are as follows:

Step 1: Firstly P(2) model is solved for each objective separately. That is

For maximization of wealth: $f_1 = Max\ W_{t+1}$ subject to the constraints (21)-(25)

For minimization of value-at-risk: $f_2 = Min\ VaR(\alpha)$ subject to the constraints (21)-(25)

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If the optimality condition is same for both the objective functions then an efficient solution is achieved. Otherwise go to step 2

Step 2: Optimum values of $f_1(f_1^+)$and $f_2(f_2^+)$are considered as upper bound and lower bound for wealth and risk objective respectively:

$$f_1^+ = f_1(x^1)$$

$$f_2^+ = f_2(x^2)$$

Step 3: To find the compromise solution, a distance function is constructed as follows

$$d = \sum_{i=1}^{2} \lambda_i (f_i - f_i^+)^2 \qquad (26)$$

Where $\lambda_i, (i = 1,2)$ reflects the relative importance of the i^{th} objective function.

Step 4: Now P(2) modelwill be represented in terms of the distance function as follows:

P(3)

$$Min\ d = \sum_{i=1}^{2} \lambda_i (f_i - f_i^+)^2$$

subject to the constraints (21-25)

V. PROPOSED NEURAL NETWORKS

This section outlines the steps taken to transform P(3) into a corresponding neural network model. First of all, to formulate P(3) in terms of the neural network, an energy function $E(z)$ is constructed in such a way that when $E(z)$ moves toward zero, z moves toward to z^*, which corresponds with the intended optimal solution. A system of differential equations is constructed by taking the gradient of an energy function that leads to the development of neural networks.

Let us first transform P(3) into standard form.

$$\begin{cases} minimize\ f(x) \\ subject\ to:\ g(x) \le 0\ , \\ h(x) = 0, \end{cases} \qquad (27)$$

where $x \epsilon \mathbb{R}^{2n}$, $g(x) = \big(g_1(x), g_2(x), \dots, g_m(x)\big)^T$

$h(x) = \big(h_1(x), h_2(x), \dots, h_l(x)\big)^T$.

Suppose that the problem (27) has an optimal solution. Let its Lagrange's function be

$$L(x, \mu, \lambda) = f(x) + \mu^T . g(x) + \lambda^T . h(x) \qquad (28)$$

Where $\mu\ = (\mu_1,\ \mu_2,\ \dots\ \mu_m)^T$ and

$\lambda = (\lambda_1,\ \lambda_2, \dots,\ \lambda_l)^T$.

Then

$$\nabla_x L(x, \mu, \lambda) = \nabla f(x) + \nabla g(x)^T \mu + \nabla h(x)^T \lambda$$

Where ∇ denotes the gradient [20] of a function.

KKT conditions for (27) are as follows

$$\begin{cases} \nabla_x L(x^*, \mu^*, \lambda^*) = 0 \\ \mu^{*T} g(x^*) = 0, \quad \mu^* \ge 0 \\ g(x^*) \le 0, \qquad h(x^*) = 0 \end{cases} \qquad (29)$$

Theorem 5.1[3]:$x^* \epsilon \mathbb{R}^{2n}$ minimizes $f(x)$ in (27) if and only if $\big(x^{*T}, \mu^{*T}, \lambda^{*T}\big)^T$, $\mu^* \epsilon \mathbb{R}^m \lambda^* \epsilon \mathbb{R}^l$satisfies (29).

$\mu^{*T}$and λ^{*T} are Lagrange's multiplier vectors and $x^*$is termed as the KKT point of (27)

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Theorem 5.2[3]: x^* is the optimal solution of (27) if and only if x^* is the KKT point of (27).

Theorem 5.3[3]: Suppose that $g(x)\epsilon\mathbb{R}^m$is the continuously differentiable function Then

$$g(x) \leq 0 \Leftrightarrow \frac{1}{2} g(x)^T [g(x) + |g(x)|] = 0 \quad (30)$$

Moreover, $\frac{1}{2} g(x)^T [g(x) + |g(x)|] \epsilon C^1$. Where |.| denotes the absolute value. Theorem 5.3, is used to transformed general inequality constraint into equality constraints, since it does not affect the differentiability of$g_j(x)$. Convexity of (30) can be easily verified when $g_j(x)$ is concave.

Similarly

$$\mu \geq 0 \Leftrightarrow \frac{1}{2} \mu[\mu - |\mu|] = 0 \quad (31)$$

With the help of (29),(30) and (31), The Energy function of (27) can be constructed as follows:

$$E(z) = E(x, \mu, \lambda) = \frac{1}{2} \|\nabla_x L(x, \mu, \lambda)\|^2 + \frac{1}{2} \|\mu. g(x)\|^2 + \frac{1}{2} \|h(x)\|^2 + \frac{1}{2} \mu[\mu - |\mu|] + \frac{1}{2} g(x)^T [g(x) + |g(x)|] \quad (32)$$

Here $\mu. g(x)$ denotes the component-wise multiplication of μ and $g(x)$.

Where $z = (x^T, \mu^T, \lambda^T)^T \epsilon \mathbb{R}^{n+m+l}$.The energy function (32) is formulated in such a way that it can't be negative i.e. $E(z) \geq 0$. So, optimality will be achieved at $E(z) = 0$.

Theorem 5.4[3]: $x^*$is the optimal solution of *(27)*$\Leftrightarrow$ $E(z^*) = 0,\ z^* = \left(x^{*T}, \mu^{*T}, \lambda^{*T}\right)^T$.

Proof: If $z^* = \left(x^{*T}, \mu^{*T}, \lambda^{*T}\right)^T$ is the zero point of $E(z)$ then it must satisfy the KKT system (29).

Hence it follows from Theorem 4, $x^*$is the optimal solution of (27).

Converse: Since from theorem 4, If x^* is the optimal solution of (27) then $z^* = \left(x^{*T}, \mu^{*T}, \lambda^{*T}\right)^T$ satisfies the KKT system (29). Now it is straightforward to show z^* is the zero point of $E(z)$.

With the help of Energy Function $E(z)$, a gradient-based neural network [3] can be formulated to solve our portfolio optimization problem.

$$\frac{dz}{dt} = -\nabla E(z)$$

Leung et al. [3] computedthe gradient in details as follows:

$$\begin{cases} \dfrac{dx}{dt} = -\nabla_x E(z) \\ \dfrac{d\mu}{dt} = -\nabla_\mu E(z) \\ \dfrac{d\lambda}{dt} = -\nabla_\lambda E(z) \end{cases} \quad (34)$$

If$\nabla E(z)$ is Lipchitz continuous, then (33) has a unique solution because Lipchitz continuity implies continuity.

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5.1 Stability analysis

The Energy function $E(z)$ (32) is bounded below, and it is easy to show that its gradient $\nabla E(z)$is locally Lipchitz continuous [3]. In gradient descent algorithm any trajectory converges to an equilibrium point and, the Lyapunov stability and asymptotical stability are identical in this context. Now it is enough to show that any trajectory of proposed approach converges to asymptotical stable point.

Theorem 5.5[3]: *Let the set of equilibrium points*$M = \{z = (x^T, \mu^T, \lambda^T)^T \, \epsilon \mathbb{R}^{n+m+l} \middle| \nabla E(z) = 0\}$of *(32)* and$\Theta = \{z = (x^T, \mu^T, \lambda^T)^T \epsilon \mathbb{R}^{n+m+l} \middle| (x^T, \mu^T, \lambda^T)^T$satisfies(29)} then$\Theta \subseteq$ M *i.e.* each point$z$that satisfies(29)must be an equilibrium point of$E(z)$.

Theorem 5.6[3]: Suppose that the neural network (33)has a unique equilibrium pointz^*, then$z^*$is uniformly and asymptotically stable.

Theorem 5.7[3]: Suppose that a level set$L(z^0) = \{z \epsilon \mathbb{R}^{n+m+l} : E(z) \leq E(z^0)\}$is boundedthen there exists an equilibrium point$\bar{z} \epsilon M$and a strictly increasing sequence$\{t_n\}, (t_n \geq 0)$such that$\lim_{n\to\infty} z(t_n, z^0) = \bar{z}, \nabla E(\bar{z}) = 0$.

Theorem 5.8[3]: Suppose that the neural network(33)has a unique equilibrium point and the level set$L(z^0) = \{z \epsilon \mathbb{R}^{n+m+l} : E(z) \leq E(z^0)\}$is bounded. Then every trajectory of network equation(32)converges toz^*that satisfies(29). *i.e.*, $z^*$is globally and asymptotically stable for(29).

Now an algorithm is used in the next section to describe the technical aspect of the proposed neural network. Algorithm starts with an initial vector at which value of energy function is calculated. Then while loop is used for further iterations. Each iteration consists of, calculating the gradient of $E(z)$, updating the initial vector, calculating the error and stopping criteria.

Algorithm:

Step 1: Initialization

let$x \epsilon \mathbb{R}^n, \mu \epsilon \mathbb{R}^m, \ \lambda \epsilon \mathbb{R}^l, \Delta t > 0, t = 0$are initial arbitrary vectors and let error $\varepsilon = 10^{-15}$.

Step 2: Construct Energy function $E(z)$ from the KKT system (29)

Step 3: Gradient of $E(z)$is calculated to obtain the system of differential equation as follows:

$$e(\mathrm{t}) = \nabla_x E(z)$$
$$v(\mathrm{t}) = \nabla_\mu E(z)$$
$$u(\mathrm{t}) = \nabla_\lambda E(z)$$

Step 4: Initial vector updation

$$x(t + \Delta t) = x(t) - \Delta t. e(t)$$
$$\mu(t + \Delta t) = \mu(t) - \Delta t. v(t)$$
$$\lambda(t + \Delta t) = \lambda(t) - \Delta t. u(t)$$

Step 5: Error calculation

$$r(t) = \sum_{i=1}^{n} e_i^2\,(t); \;\; l(t) = \sum_{j=1}^{m} v_j^2\,(t); \;\; m(t) = \sum_{k=1}^{l} u_k^2\,(t)$$

Step 6: if r, l and $m < \varepsilon$, stop; else $t = t + 1$

end

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Output: $x(t+\Delta t),\ \mu(t+\Delta t),\ \lambda(t+\Delta t)$

VI. NUMERICAL ILLUSTRATION

An empirical study is conducted in the Bombay Stock Exchange (BSE) market to appraise the designed model for multi-period portfolio optimization. Suppose that an investor chooses to invest in ten risky assets from different sectors (e.g., the automobile, banking and finance, cement and construction, chemical, and manufacturing sectors). The mean and value-at-risk for the different confidence levels of stocks are organized in Table 1, and the data associated with the annual turnover rate are presented in Table 2. The initial wealth of the investor is considered as one unit and can be modified in the subsequent period. The return rates of stocks have been assumed to be a zigzag uncertain variable. One year's worth of data (from January 2017 to December 2017) for these stocks are used for subsequent calculations. The transaction expenses are fixed at 0.2% of the turnover rate.

To solve the P(3) model, a gradient-based neural network technique is employed to simulate the dynamic system. Calculations have been made under various uncertain confidence levels (β); the results are presented in Table 3. It is evident that the value-at-risk and wealth objective have opposite trends. Hence, to achieve one objective, an investor has to compromise the other.

Table 1 Mean and Value at Risk of monthly returns of risky asset

S.No.	Stock Code	Mean	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.2$
1	500180	0.0422	0.0044	0.0040	0.0032
2	500209	-0.0039	0.0063	0.0051	0.0029
3	500312	-0.0094	0.0043	0.0033	0.0017
4	500425	0.0221	0.0053	0.0045	0.0030
5	500470	0.0368	0.0135	0.0117	0.0084
6	500495	0.0855	0.0365	0.0315	0.0226
7	500650	0.0548	0.0260	0.0221	0.0151
8	503806	0.0091	0.0067	0.0055	0.0034
9	506480	0.0779	0.0246	0.0214	0.0157
10	514034	-0.0380	0.1147	0.0912	0.0522

Table 2 Annual Turnover rates of the assets in different confidence level

S.No.	**Stock Code**	**Mean**	$\beta = 0.7$	$\beta = 0.8$	$\beta = 0.9$
1	500180	0.0025	0.00176	0.00144	0.0032
2	500209	0.0056	0.00434	0.00396	0.0029
3	500312	0.009	0.00563	0.00472	0.0017
4	500425	0.0094	0.00521	0.00424	0.0030
5	500470	0.0846	0.04828	0.03492	0.0084
6	500495	0.1575	0.105	0.09	0.0226
7	500650	0.01405	0.00747	0.00568	0.0151
8	503806	0.02062	0.0123	0.0102	0.0034

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9	506480	0.0096	0.00367	0.00248	0.0157
10	514034	0.0250	0.01266	0.01004	0.0522

Table 3.1 The results of multi-objective portfolio optimization under different risk and liquidity confidence levels

Alpha Value	Beta Value	Stock Code	Proportion
		500180	0.3000
		500312	0.0500
0.05	0.7	500495	0.1409
		500650	0.2561
		503806	0.2529
		500180	0.3000
		500209	0.0500
0.05	0.8	500312	0.3000
		500425	0.0500
		500495	0.3000
		500180	0.3000
		500209	0.0968
0.05	0.9	500312	0.3000
		500495	0.2498
		506480	0.0534

Table 3.2

Alpha Value	Beta Value	Stock Code	Proportion
		500425	0.0500
		500470	0.3000
0.1	0.7	500650	0.3000
		503806	0.2737
		506480	0.0763
		500180	0.1306
		500209	0.0804
0.1	0.8	500312	0.3000
		500495	0.1889
		500650	0.3000
		500180	0.0500
		500209	0.1594
0.1	0.9	500312	0.3000
		500495	0.2473
		506480	0.2433

Table 3.3

Alpha Value	Beta Value	Stock Code	Proportion
		500180	0.3000
0.2	0.7	500209	0.1410
		500495	0.1513

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		500650	0.3000
		514034	0.1077
0.2	0.8	500180	0.1602
		500209	0.0500
		500312	0.3000
		500495	0.1898
		500650	0.3000
0.2	0.9	500180	0.0500
		500209	0.1009
		500312	0.3000
		500495	0.2491
		506480	0.3000

Table 4 Final Risk, Return and Liquidity of multi-objective portfolio optimization under different risk and liquidity confidence levels

S.no.	Alpha value	Beta value	Return	Risk	Liquidity
1.	0.05	0.7	0.0406	0.015	0.0206
2.	0.05	0.8	0.03641	0.0141	0.0342
3.	0.05	0.9	0.035	0.0136	0.0291
4.	0.1	0.7	0.03702	0.016	0.0206
5.	0.1	0.8	0.035	0.01704	0.0243
6.	0.1	0.9	0.03879	0.01751	0.0293
7.	0.2	0.7	0.03741	0.0279	0.0206
8.	0.2	0.8	0.03644	0.01701	0.0244
9.	0.2	0.9	0.04357	0.0186	0.0295

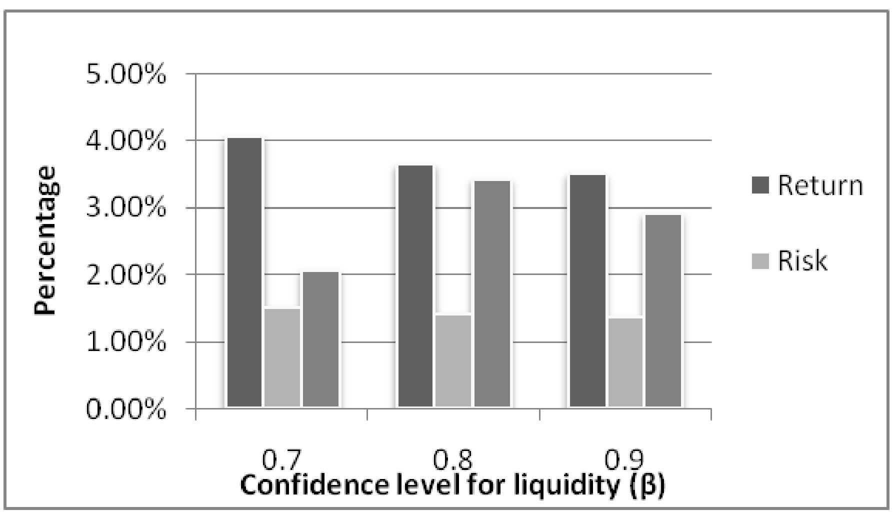

Figure 2. Comparison chart for $\alpha = 0.05$

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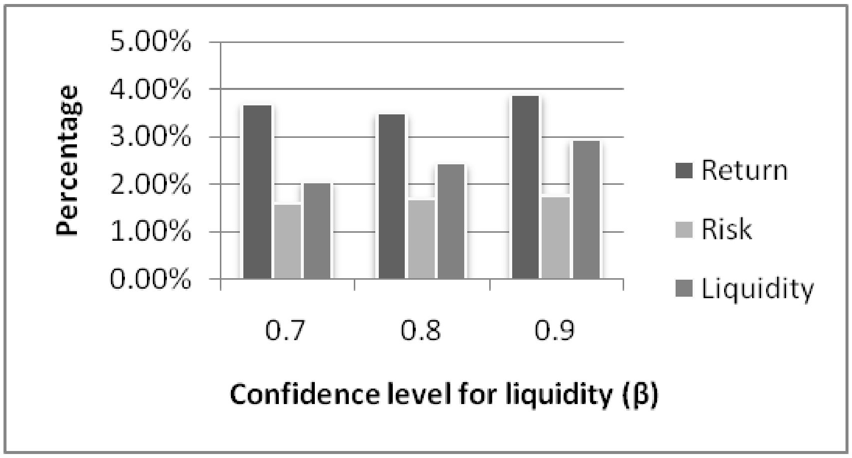

Figure 3. Comparison chart for $\alpha = 0.1$

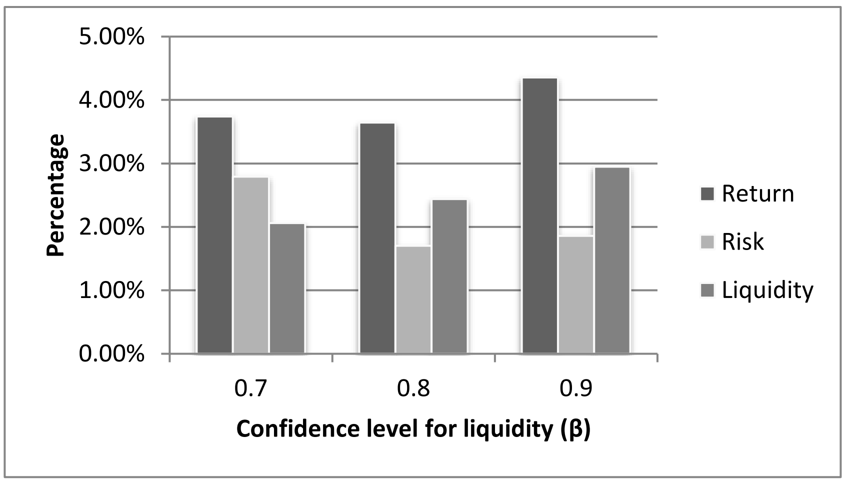

Figure 4 Comparison chart for $\alpha = 0.2$

Table 5 Wealth for consecutive three periods under different risk and liquidity confidence interval $(W_1 = 1)$

S.no.	Alpha value	Beta value	W_2	W_3	W_4
1.	0.05	0.7	1.0385	1.0786	1.1203
2.	0.05	0.8	1.0344	1.0701	1.1068
3.	0.05	0.9	1.0329	1.0670	1.1023
4.	0.1	0.7	1.0350	1.0713	1.1088
5.	0.1	0.8	1.0329	1.0670	1.1023
6.	0.1	0.9	1.0367	1.0749	1.1144
7.	0.2	0.7	1.0354	1.0721	1.1100
8.	0.2	0.8	1.0344	1.0701	1.1069
9.	0.2	0.9	1.0415	1.0849	1.1300

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VII. CONCLUSION

This paper focuses on a portfolio optimization model by considering more than one period and by considering transaction costs in uncertain situations. The optimization problem is developed as a bi-objective problem. Wealth is chosen as one objective and value-at-risk as another. Liquidity, bounds, and cardinality are the constraints imposed on the model. An artificial neural network algorithm is adapted to achieve optimal results. The results are computed for different values of α (VaR confidence level) and β (liquidity confidence level) so that most investors' aspirations and preferences are accounted for, which will help them achieve their desired outputs.

In the stock market, historical data alone are insufficient for one to make predictions about the future; therefore, uncertainty theory is more suitable than other theories for handling the subjectivity inherent in imprecise market situations. Furthermore, a neural network as an artificial intelligence method is a tool that allows us to solve very sophisticated and complex optimization problems that would otherwise require the use of several optimization techniques to reach optimal solutions. Thus, uncertainty theory saves significant time and effort.

In the future, we will attempt to incorporate more parameters, like skewness, kurtosis, entropy, etc., as objective functions and conditional value-at-risk (CVaR) and conditional downside risk (CDaR) as risk measures to improve the comprehensiveness of the optimization model. We also intend to introduce other uncertainty variables, like a normal uncertain variable and an empirical uncertain variable, into the portfolio optimization model and to contrast the outcomes of these models with those derived from the present analysis.

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