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Some Basic Principles on Posets, Hasse Diagrams and Lattices

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Abstract: In this paper we give some important definitions, examples and properties of partly ordered sets or simply a poset, Hasse diagrams and Lattices.

Keywords: *Partial ordered set,hasse diagram, lattices etc.*

I. Introduction

In 1963Gabor Szasz [1], was introduced in Introduction to Lattice Theory definitions and some examples and in 2010 S. Santha [2] gave some more examples of posets, hasse diagrams and lattices. In 2013, V. B. V.N. Prasad & J V Rao, were published Characterization of Quasi groups and Loops [3], Classification of partially ordered loops and lattice ordered loops [4], in 2014 they were published Classification of "Normal sub loop and Ideal of loops" [5] and in 2014 Cones in Lattice ordered loops [6] were published in lattice ordered loops.

II PARTIAL ORDERE RELATION

Definition2.1:"*A partly ordered set is a system comprising of a non-empty set P and a binary relation ≤ on P such that the following conditions are satisfied* $\forall$ *x, y, z*$\in$*P*".

(i) "Reflexive: $x\leq x$
(ii) Anti-symmetric: $x\leq y$ and $y\leq x \Rightarrow x=y$
(iii) Transitive: $x\leq y$ and $y\leq z \Rightarrow x\leq z$ "

Note2.1: We call the relation "≤(less than or equal to)"is a partial order on the set P and P is said to be "a partly Ordered set "or a "Partially Ordered set" or simply a "Poset" by the relation "≤".

Note 2.2: It is easy to observe that if ≤ is a partial order on P then≥ is also a partial order on P and we call the partly ordered set (P, ≥) the dual of the partly ordered set (P,≤).

Definition 2.2:"Let (P, ≤) be a poset, the elements a, b$\in P$ are said to be 'comparable' if either $a\leq b$ or $b\leq a$. Otherwise they are called 'incomparable' elements".

Definition 2.3: If (P, ≤) is a poset and the elements a, b$\in P$ are comparable then (P,≤) is called a chain.

Definition2.4: Let R be a nonempty subset of a partly ordered set P. An element a$\in P$ is known as an upper bound of R, if $x\leq a\ \forall x \in R$. If R has at least one upper bound then we say that R is bounded above in P.

Definition2.5: An upper bound a of R is said to be a (LUB) least upper bound of R if, for some upper

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bound b of R a$\leq b$ (i.e. least among the upper bounds). The least upper bound of R in P is denoted by lub$_P$R or Sup$_P$R (supremum of R).

Note2.3: Any sub set R of P has at most one least upper bound in P.

Definition2.6: Let R be a nonempty sub set of a Poset P. An element a$\in P$ is called a lower bound of R if a$\leq$x $\forall x \in R$. If R has at least one lower bound then we say that R is bounded below in P.

Definition: 2.7"A lower bound a of R is called a (GLB) greatest lower bound of R, for any lower bound b of R b$\leq$a (i.e. greatest among the lower bounds). The greatest lower bound of R is denoted by g.l.b.$_P$R or Inf$_P$R (infimum of R) or simply g.l.b.R or inf R".

Note 2.4: R can have almost one g.l.b in P.

Definition2.8: Let (P, $\leq$) *be a poset*, an element a$\in P$ is called a least element of Pif a $\leq$ x $\forall x \in P$. Clearly P has at most one least element and when it exists, it is denoted by 0 (zero). Clearly 0=inf P.

Definition 2.9: Let (P,$\leq$) *be a poset*, an element a $\in P$ is called a greatest element of P if x$\leq$a$\forall x \in P$.

Clearly P has at most one greatest element and when it exists, it is denoted by 1 (one) and clearly 1 SupP.

Definition 2.10:"Suppose (P, $\leq$) is a poset and a, b$\in$P $\ni$ a<b. Then we say that a is covered by b or b covers a if $\exists$ x$\in$P such that a<x<b".

III HASSE DIAGRAMS

Definition 3.1:"A finite poset P can be signified by means of a figure in the subsequentmethod. Signifyindividuallyan element a $\in$P by a small circle $k_a \ni$ whenever a<b, then k_b is higher than k_a in the diagram. Additional join k_a and k_b by a straight line whenever b covers a".

If b>a in a finite poset iff there is a sequence a=a_1,......a_n=b such that each a_{i+1} covers a_i.

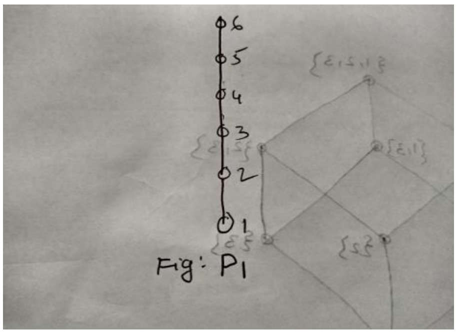
Thus a<b iff there is an ascending brocken line connecting a to b.

If no line connects a and b$\neq a$ then a and b are in comparable.

The resulting figure is the diagram of the partly ordered set P w.r.t. the given ordering and this is called the Hasse diagram of P.

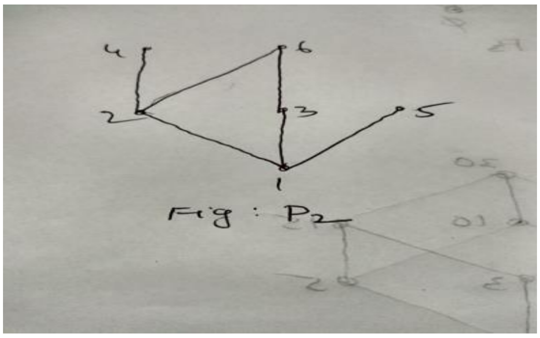
Example 3.1: Let P_1= {1, 2, 3, 4, 5, 6} ordered by the usual relation "less than or equal to". Draw the Hasse diagram.

Sol:



Example 3.2: Draw the Hasse diagram of P_2== {1, 2, 3, 4, 5, 6} ordered by divisibility.

Sol:



Example 3.3: For the poset P_3=P ({1, 2, 3}) ordered by set inclusion, write the elements in P_3 and also draw the Hasse diagram.

Sol: Given that P_3=P ({1, 2, 3})

P ({1, 2, 3}) = {ϕ, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}}

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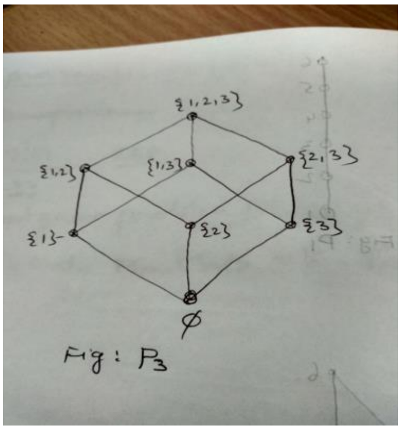
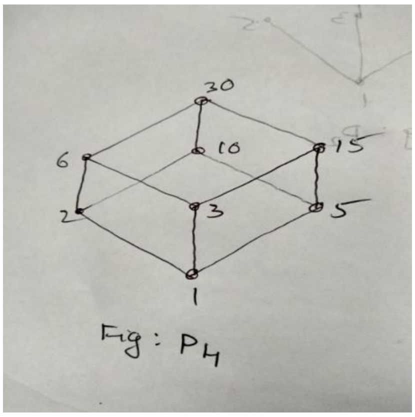


Example 3.4: For the set of all divisors of 30 with respect to the usual divisibility, draw the Hasse diagram.

Sol:



IV LATTICES

Definition 4.1: "A Lattice is a Poset (L, ≤) in which each pair of elements has a"(LUB) Least Upper Bound and a (GLB) Greatest Lower Bound" .

The least upper bound (or supremum, l.u.b.) of a,bϵL is denoted by a⨁b, a$\vee b, a \cup b$ or $a + b$ and is called the join of a and b.

The (GLB) greatest lower bound (or infimum) of a, b∈L is denoted by a*b, a$\wedge b, a \cap b$ or $a.b$ and is called the meet of a and b.

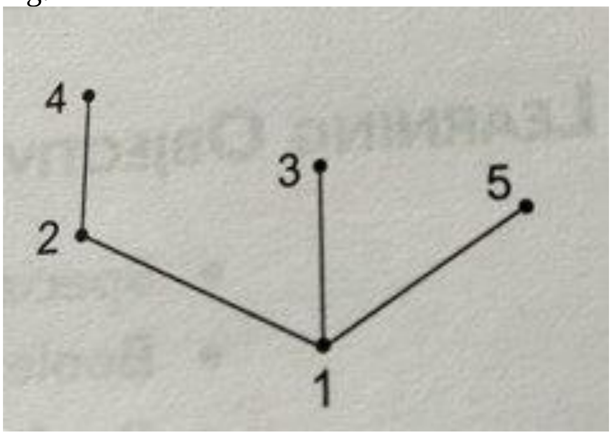
Note 4.1: $\vee$ and $\wedge$ are binary operations on a lattice, since the l.u.b. and the g.l.b. of any subset of a poset are unique.

Note 4.2:"Every lattice is a Poset but every Poset is not a lattice".

Example 4.1: Verify that the following posets are lattices or not (i) ({1,2,3,4,5},/) and (ii) ({1,2,4,8,16},/).

Solution: Consider the poset ({1,2,3,4,5}, /).

The corresponding Hasse diagram is shown in Fig.1



For the pairs of elements (2, 3) and (3, 5), there is no upper bound. Hence, the least upper bound does not exist.

This implies that ({1, 2, 3, 4, 5}, /) is not a lattice.

So we can say that all the partially ordered sets are not lattices.

(ii) Consider the Poset ({1, 2, 4, 8, 16}, /). The corresponding Hasse diagram is shown in Fig.2

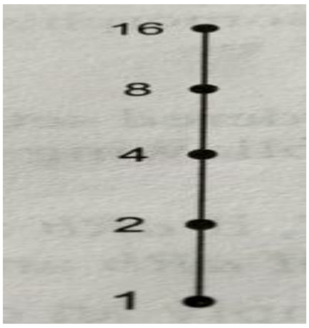


Fig.2

l.u.b.of {1,2}=2	g.l.b.{1,2}=1
l.u.b. of {2, 4}=4	g.l.b. {2,4}=2
l.u.b. of {4, 8}=8	g.l.b. {4, 8}=4

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l.u.b. of {2, 8}=8	g.l.b. {2, 8}=2
l.u.b. of {8, 16}=16	g.l.b. {8, 16}=8
l.u.b. of {4, 16}=16	g.l.b. {4,16}=4
l.u.b. of {2, 16}=16	g.l.b. {2,16}=2
l.u.b. of {1, 4}=4	g.l.b. {1,4}=1
l.u.b. of {1, 8}=8	g.l.b. {1, 8}=1
l.u.b. of {1, 16}=16	g.l.b. {1, 16}=1

Therefore, each pair of elements of the poset has a l.u.b. and a g.l.b.

Hence, the poset ({1, 2, 4, 8, 16}, /) is a lattice.

Example 4.2: If P(S) is the power set of a set S,$\cup$ and $\cap$ are taken as the join and meet, then prove that {P(S), $\subseteq$} is a lattice.

Definition 4.2: Any statement about lattices involving the operations $\wedge\, and\, \vee$ and the relations $\leq and\ \geq$ remains true if $\wedge$ is replaced by $\vee$ and $\vee$ is replaced by $\wedge$, $\leq by\ \geq and\ \geq by\ \leq$.The lattices (L, $\leq$) $and\ (L, \geq)$are called duals of each other. Similarly, the operations $\vee$ and $\wedge$ are duals of each other and the relations $\leq$ and $\geq$ are duals of each other.

Example 4.3: Determine whether the principle of duality is followed in the poset ({1, 2, 4, 8, 16},·).

Solution: Consider the Hasse diagram given in Fig.4 for the poset ({1, 2, 4, 8, 16}, ·).

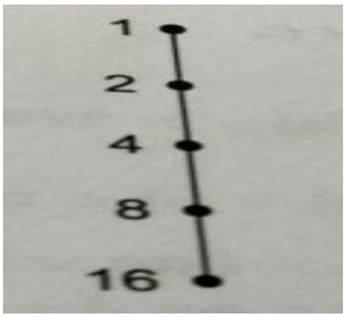


Fig.4

Let S= {1, 2, 4, 8, 16} and let A$\subseteq$ S.

Then, the l.u.b. of A with respect to $\leq$ is equal to the g.l.b. of A with respect to $\geq$ and vice versa.

Thus, when $\leq and\ \geq$ are exchanged, the (LUB). and the (GLB) are exchanged.

If "(L, $\leq$) is a lattice, then (L, $\geq$) is also a lattice". Also, the operations of "join and meet on (L, $\leq$) become the operations of meet and join respectively, on (L, $\geq$)".

Thus, the principle of duality is followed in the poset ({1, 2, 4, 8, 16},·).

Example 4.5: Let (L, $\leq$) be a lattice and a, b, c$\in$ L. If a$\leq$c, b$\leq$c, then prove that a$\vee$ $b \leq c\ and\ a \wedge b \leq c$

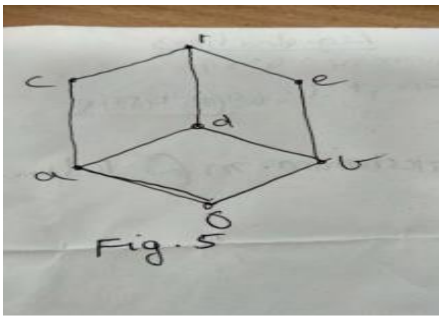
Diagrams of Lattices:

Example (1): Let L= {0, a, b, c, d, e, 1}, where $0\wedge x = 0, 0 \vee x = x, 1 \wedge x = x, 1 \vee x = 1\ and$

$\wedge$	a	b	c	d	e
a	a	0	a	a	0
b	0	b	0	b	b
c	a	0	c	a	0
d	a	b	a	d	b
e	0	b	0	b	e

$\vee$	a	b	c	d	e
a	a	d	c	d	1
b	d	b	1	d	e
c	c	1	c	1	1
d	d	d	1	d	1
e	1	e	1	1	e

Then this lattice can be represented by the diagram

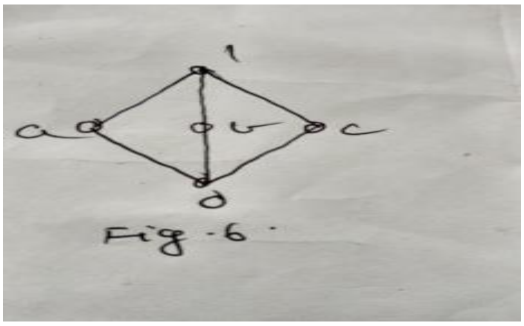


Example (2): Draw the diagram of the lattice for the following table L={0, a, b, c, 1} 0<a, b, c<1

∧	a	b	c
a	a	0	0
b	0	b	0
c	0	0	c

∨	a	b	c
a	a	1	1
b	1	b	1
c	1	1	c

Then this lattice can be represented by the diagram



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