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An Innovative Study on Lie Groups and Lie Algebras

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Abstract: This research article mainly explores on matrix Lie groups admitting the Cayley Construction and presents innovative proofs of the following propositions.

- If a matrix group admits the Cayley construction and so is a matrix Lie group then the corresponding vector space coincides with the Cayley image of it.
- Every matrix Lie group possesses an in-image.

Furthermore three most important lemmas and one proposition in Lie Groups and Lie Algebras are presented with very simple and innovative proofs. One of three lemmas gives the necessary and sufficient condition for a topological group to be Hausdorff.As well the condition for a topological group to be connected is also derived.

Keywords:*Matrix Lie group, embedding, smooth, Cayley image orthogonal sympletic group, analytic diffeomorphism. Smooth group, morphism, manifold, Hausdoff, smooth mapping, Lie group.*

I. INTRODUCTION

L.D. Faddeer et.al [1] in 1988, in their research paper discussed quantum formal groups, a finite dimensional example and reviewed the deformation theory and quantum groups. J.C. Benjumea et.al [2], in 2005, in their research article presented a method to obtain the Lie group associated with finite dimensional nilpotent Lie algebra. In 1968, J.G. Belinfante et.al [3], in their research paper classified the finite dimensional representations of semi simple Lie algebras. David A. VoganJr, in 1979, in his research paper presented an essentially algebraic description of the irreducible representations of connected semi simple Lie group G in terms of their restriction to a maximal compact subgroup K of G. In 1947, Claude Chevalley [4], in his research article discussed the applications which can be made of the notion of algebraic Lie algebra to the general theory of Lie algebra and particularly of semi-simple algebra.

A subgroup H of $HK(m)$ is called a matrix Lie group ($HK(m)$ is full linear group) if a smoothness is introduced on H w.r.t which it is a Lie group and the embedding $\sigma : H \to HK(m)$ is smooth and hence it is a homomorphism of a Lie groups. Every one parameter subgroup of the group H is automatically a one parameter subgroup of $HK(m)$ and so is of the form $s \to e^{sB}$. This defines a 1-1 function $I(H) \to I(HK(m)) = \square(m)$ which is nothing but the mapping $I(\sigma)$. Thus any matrix Lie group the vector space $g = I(H)$ is naturally identified with some subspace of the vector space $\square(m)$. Any group admitting the Cayley construction is an example of matrix Lie group say a group $E_D(m)$ of all D-orthogonal matrices. By definition a matrix one-parameter subgroup $s \to e^{sB}$ is a one-parameter subgroup of $E_D(m)$ iff for any $s \in \square$, $(e^{sB})^z De^{sB} = D$, w.r.t s differentiate this and

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put $s=0$,one can get $B^2D+DB=0$ .Using definition this relation provides that B is a D-skew symmetric matrix. Conversely the function $s\to e^{sB}$ is seen to be a one-parameter subgroup of $E_D(m)$ for any D skew symmetric matrix B. To prove this one can use the analogue of the family relation $e^b=\lim_{q\to\infty}\left(1+\frac{b}{q}\right)^m$ one can prove that this formula is valid in any finite dimensional associate algebra.

In fact since $\frac{1}{q^l}\binom{q}{l}=\frac{q(q-1)----(q-k+1)}{q.q.q.................q(ltimes)}\frac{1}{\lfloor s}\le\frac{1}{\lfloor l}$

For any multiplicative norm

$$\left\|e^b-\left(e+\frac{b}{q}\right)^q\right\|\sum_{l=0}^{\infty}\left(\frac{1}{\lfloor l}-\frac{1}{q^l}\binom{q}{l}\right)b^l$$

$$\le\sum_{l=0}^{\infty}\left(\frac{1}{\lfloor l}-\frac{1}{q^l}\binom{q}{l}\right)\|b\|^l$$

$$=e^{\|b\|}-\left(1+\frac{\|b\|}{q}\right)^q$$

$$\lim_{q\to\infty}\left\|e^b-\left(e+\frac{b}{q}\right)^q\right\|=0 \text{ as}\left(1+\frac{\|b\|}{q}\right)^q\to e^{\|b\|}$$

One can see that for any s,

$$De^{sB}=D\lim_{q\to\infty}\left(F+\frac{sB}{q}\right)^q=\lim_{q\to\infty}D\left(F+\frac{sB}{q}\right)^q$$

$$=\lim_{q\to\infty}\left(F+\frac{sB^z}{q}\right)^q D=e^{-sB^z}D$$

Since $Dh(B)=h\left(-B^z\right)D$ for any polynomial $h(B)$ of matrix B.

So $\left(e^{sB}\right)^z De^{sB}=\left(e^{sB}\right)^z e^{-sB}D=D$.

Hence in fact $\left(e^{sB}\right)\in E_D(m)$.

This prove that for a group $E_D(m)$ the vector space of all D-skew symmetric matrices is a subspace $I\left(E_D(m)\right)$ of space $\square(m)$ and the subspace $I\left(E_D(m)\right)$ coincides with the Cayley image of $E_D(m)$

A group M is called a lie group or smooth group if the functions

$f:M\times M\to M$ defined by $f(a,b)=ab$ ……..(1)

and $f:M\to M$ defined by $f(a)=a^{-1}$ ………..(2) are smooth functions. If M and N are lie groups then the mapping $M\to N$ is called morphism of lie groups if it is their homomorphism as abstract groups and their smooth mapping as manifolds. All lie groups and all their homeomorphisms form a category devoted by GR-DIFF. Depending on D's smoothness there is a countable family of categories GR-DIFF where either $2\le s<\infty$ or $s=\infty$ one require of the manifolds under consideration, but practically nothing depends on s as any D's-isomorphic to the analytic class group. Some researchers find slight differences between Lie groups and smooth groups. The group M which is at the same time a topological space is called a topological group if the functions (1) and (2) are continuous for it. The homomorphism $M\to N$ of topological groups is called continuous if it is a continuous mapping. Topological groups and their continuous homomorphism are called the category GR-TOP. The topological space P is said to be Hausdorff or separable if any two of its different points have disjoining neighbourhoods. In other words the diagonal ∇ is called in P. A topological group should not necessarily be Hausdorsff.

II. PROPOSITION

If the matrix group $H\subset HK(m)$ admits the Cayley construction and so is a matrix Lie group then the corresponding vector space $I(H)$ coincides with the Cayley image H^* of H.

Proof: Let $B\in I(H)$ that is , let the mapping $s\to e^{sB}$ be a one-parameter subgroup of the group

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H Since the set H^* of non-exceptional matrices in H is a neighbourhood of the identity F of H there is $\varepsilon > 0$ such that $|s| < \varepsilon$ the matrix e^{sB} is non-exceptional and therefore its Cayley image

$\left(e^{sB}\right)^* = \left(F - e^{sB}\right)\left(F + e^{sB}\right)^{-1} \in H^*$ is defined. Since H^* is a vector space it follows that the matrix

$\left[\dfrac{d\left(e^{sB}\right)^*}{ds}\right]_{s=0} = \lim\limits_{s\to 0}\dfrac{\left(e^{sB}\right)^*}{s}$ also belongs to H^*. But on the other hand

$$\left[\frac{d\left(e^{sB}\right)^*}{ds}\right] = Be^{sB}\left(F + e^{sB}\right)^{-1} + \left(F - e^{sB}\right)\frac{d\left(\left(F + e^{sB}\right)^{-1}\right)}{ds}$$

And $\left[\dfrac{d\left(e^{sB}\right)^*}{ds}\right]_{s=0} = \dfrac{1}{2}B$ and consequently $B \in H^*$

III. PROPOSITION

A subgroup $HK(m)$ is a matrix Lie group if there is a diffeomorphism $h: U \to \overset{\circ}{U}$ of some neighbourhood U of the identity matrix in $HK(m)$ onto an open set $\overset{\circ}{U}$ of $\square(m)$ that has the property that the set $h(H \cap U)$ is the intersection of H^* and some vector subspace H^* of the space $\square(m)$.

$$h(H \cap U) = H^* \cap \overset{\circ}{U}$$

Proof: Let $q = \dim H^*$ and let $\varphi: H^* \to \square^q$ be an isomorphism of the space H^* onto the space $\square^q$. Also let $W = H \cap U$ and $\overset{\circ}{W} = \varphi\left(H^* \cap \overset{\circ}{U}\right)$. Then $\overset{\circ}{W}$ is an open set in $\square^q$ and the mapping $k = \varphi \circ h$ onto W is a 1-1 onto correspondence $W \to \overset{\circ}{W}$. In other words the pair (W, k) is a chart on H.

Now let B be an arbitrary matrix in H and let $W_B = K_B(w)$ and $k_B = k \circ k_B^{-1}$. Then the pair (W_B, k_B) is also chart on H. Since $B \in W_B$, all sets of the form W_B over H. Moreover if $W_B \cap W_C \neq 0$ then on $k_B\left(W_B \cap W_C\right)$ the mapping $k_C \circ k_B^{-1}$ will be a restriction of the diffeomorphism

$$k \circ k_C^{-1} \circ k_B \circ k^{-1} = \varphi \circ h \circ k_{C^{-1}B} \circ h^{-1} \circ \varphi^{-1}$$

and hence it will itself be a diffeomorphism. Consequently charts (W_B, k_B) make up an atlas. This defines on H smoothness w.r.t which H is obviously a matrix Lie group.

IV. PROPOSITION:

If for a subgroup H of a group $HK(m)$ there is a analytic diffeomorphism $h: U \to \overset{\circ}{U}$ which satisfies the conditions of proposition 3, then the vector space $I(H)$ corresponding to that group coincides with the vector space H^* specified in proposition 3.

Proof: Let $s \to e^{s\beta}$ be an arbitrary one –parameter subgroup of a group H and let $\varepsilon \to 0$ be a number such that for $|s| < \varepsilon$ the matrix $e^{s\beta}$ belongs to U then

$e^{s\beta} \in H \cap U$ and hence $h\left(e^{s\beta}\right) \in H^* \cap \overset{\circ}{U}$.

Therefore $\dfrac{dh\left(e^{s\beta}\right)}{ds} \in H^*$ .But $\left[\dfrac{dh\left(e^{s\beta}\right)}{ds}\right]_{s=0} = \left[h^1\left(e^{s\beta}\right)Be^{s\beta}\right]_{s=0} = b_1 B$ as

$h^1(z) = b_1 + 2b_2(z-1) + \ldots\ldots\ldots + qb_q(z-1)^{q-1} + \ldots\ldots.$ and when $h^1(F) = b_1 F$ .Consequently $b_1 B \in H^*$ and so $B \in H^*$ for under the hypothesis $b_1 \neq 0$. This proves that $I(H) \subset H^*$. So $I(H) = H^*$ since these vector spaces are of the same dimension.

V. PROPOSITION:

Every matrix Lie group H possesses an $\ln$ –image

Proof: According to the foregoing the only candidate for the role of the vector space H^p is the vector space $I(H)$. One can show that it in fact the necessary property. Suppose as before that U and $\overset{\circ}{U}$ are neighbourhoods of the identity and zero matrices respectively such that the function $B \to \ln B$ defines a diffeomorphism $\ln : U \to \overset{\circ}{U}$ with the inverse diffeomorphism $\exp : \overset{\circ}{U} \to U$. Then for any matrix $B \in I(H) \cap \overset{\circ}{U}$. There is an inclusion $e^B \in H \cap U$ as $e^{sB} \in H$ for any s. Since $\ln e^B = B$ this proves that $I(H) \cap \overset{\circ}{U} \subset \ln(H \cap U)$. Conversely let $C \in H \cap U$. Then a matrix $B = \ln C \in \overset{\circ}{U}$ is defined. Consider on $HK(m)$ the corresponding left invariant vector field $P : Y \to YB$. The restriction $Q = P/H$ of the field P to H is obviously a smooth left-invariant vector field on H (an element of the vector space $I(H)$) which is σ connected with the field P. Where $\sigma : H \to HK(m)$ is an embedding. This means that $I(\sigma)Q = P$. Consequently by the virtue of the general identifications the field Q identified with the matrix B. Hence $B \in I(H)$. This proves that $\ln(H \cap U) \subset I(H) \cap \overset{\circ}{U}$

Thus $\ln(H \cap U) = I(H) \cap \overset{\circ}{U}$.

6. Lemma:A topological group M is Hausdorffiff its identity is closed.

Proof: Any point in Hausdorff space is closed and hence the condition is required. But as the diagonal $\nabla \subset M \times M$ is the inverse image of the identity under the continuous function $M \times M \to M$, $(x, y) \to xy^{-1}$ it is also enough. From this lemma one can obtain that every Lie group is a Hausdorff topological group. In defining smooth groups the condition that function (2) must be smooth is required.

7. Lemma: Let A, B, C be smooth manifolds and let $\theta : A \times C \to B$ be a smooth mapping such that for any point $l \in C$ the function $\theta_i : A \to B, t \to \theta(t,l), t \in A$ is a diffeomorphism of A onto the manifold B then the function $\eta : B \times C \to A$ given by $\eta(q,l) = \theta_l^{-1}(q)$, where $q \in B, l \in C$ is a smooth mapping.

Proof:

The mappings $F : A \times C \to B \times C, G : B \times C \to A \times C$ be defined respectively by $F(t,l) = (\theta(t,l),l) = (\theta_l(t),l), t \in A, l \in C$ and $G(q,l) = (\eta(q,l),l) = (\theta_l^{-1}(q),l), q \in B, l \in C$ In this obvious that these functions are smooth iff so are the functions θ, η respectively. So under the hypothesis the mapping F is smooth and it is required to prove that so is the function G.To this end one can observe by definition $(GoF)(t,l) = G(\theta_l(t),l) = (\theta_l^{-1}(\theta_l(t),l)) = (t,l)$. For any point $(t,l) \in A \times C$. In the same way

$$(FoG)(q,l) = F(\theta_l^{-1}(q),l)$$
$$= (\theta_l(\theta_l^{-1}(q),l),l) = (q,l)$$

For any point $(t,l) \in B \times C$. This is to say that the functions F and G are inverse to each other and hence both are one - one and onto correspondences. The statement about that the smoothness of G therefore is equivalent to the statement that the smooth bijective function F is a diffeomorphism. However it is evident that a smooth one-one and onto correspondence is a diffeomorphism. Everything has thus boiled down to calculating at each point $(b,l) \in A \times C$ the differential $(dF)_{(b,l)}$ of the mapping F which can be identified as a linear function of the form

$$Z_b(A) \oplus Z_l(C) \to Z_c(B) \oplus Z_l(C) \text{..................} (3)$$

where $c = \theta(b,l)$. Every function (3) is given graphically by a matrix of the form $\begin{pmatrix} H & \mathrm{I} \\ \mathrm{J} & \mathrm{K} \end{pmatrix}$........

(4)

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Where $H : Z_b(A) \to Z_c(B), I : Z_l(C) \to Z_c(B)$, $J : Z_b(A) \to Z_l(C)$ and $\mathrm{K}:\mathrm{Z}_l(C) \to Z_l(C)$ are linear mapping defined in clear manner. In particular for the mapping $(dF)_{(b,l)}$ the mapping H is nothing but the differential at a point 'a' of the mapping $\theta_l : A \to B$, the mapping and J is the differential of the constant mapping and consequently it is a zero mapping and the mapping K is the differential of the identity mapping and so it is also an identity mapping. Hence for the differential $(dF)_{(b,l)}$ matrix (4) is of the form $\begin{pmatrix} (d\theta_l)_b & \mathrm{I} \\ 0 & \mathrm{id} \end{pmatrix}$. Since the differential $(dF)_{(b,l)}$ is an isomorphism by the hypothesis of the lemma it follows that the differential $(dF)_{(b,l)}$ is also an isomorphism. For every group M any element $b \in M$ defined by the formulas $P_b y = by, S_b y = yb, y \in M$ and two formulas

$P_b : M \to M, S_b : M \to M$ which are called shifts by an element b (the function P_b is called a left shift and the function S_b is a right shift). The following propositions of shifts are clear.

$P_f = S_f = id$ where f is the identity in M .

$P_c \circ P_b = P_{cb}, S_c \circ S_b = S_{bC}, P_b \circ S_b = S_c \circ P_b$ .Since $P_b \circ P_b^{-1} = P_{b^{-1}} \circ P_b = P_f = id$ and $S_b \circ S_b^{-1} = S_{b^{-1}} \circ S_b = S_f = id$. In particular one can see that every shift is a 1-1 onto function with

$P_b^{-1} = P_{b^{-1}}, S_b^{-1} = S_{b^{-1}}$ for any element $b \in M$. If M is a topological (smooth) group then the functions P_b and S_b are continuous (smooth) and so they are homeomorphisms.

8. Proposition: If for a group M which is at the same time a smooth manifold function (1) is smooth then so is function (2) and hence the group M is a Lie group.

Proof:The smoothness of mapping (1) implies the smoothness of shifts P_b and hence the fact that they are diffeomorphisms. The corresponding function $P : (y,b) \to P_b(y) = by$ is nothing but mapping (1) and is therefore smooth. Thus under the hypothesis of lemma (1) (for A=B=C=M) and consequently by this lemma the mapping $P' : M \times M \to M$ defined by the formula

$P^1(y,b) \to P_b^{-1}(y) = b^{-1}y$ is smooth. To complete the proof it remains to notice that the mapping $x \to x^{-1}$ is the composition of the smooth mapping $M \to M \times M$, $x \to (f,b)$ and of the mapping P^1. Therefore it is also smooth.

5. OBSERVATIONS:

(i) Any abstract discrete topological group is a Lie group as a zero-dimensional smooth manifold.

(ii) Any finite dimensional vector space is a Lie group under addition.

(iii) A unit circle $|z| = 1$ whose points are complex numbers $z = e^{i\theta}$ is a Lie group under multiplication.(iv)The discrete product $U \times V$ of two smooth (or topological) groups $U and V$ is a smooth (respectively topological) group.

(v) Any torus, $T^n, n \geq 1$ is a Lie group

(vi)A full linear group is a Lie group

(vii) The intersection $Sp(m;\square) \cap O(2m)$ is called an orthogonal sympletic group. The Cayley images of non-exceptional matrices of this group are of the form $\begin{pmatrix} A & \mathrm{B} \\ \text{-B} & \mathrm{A} \end{pmatrix}$. Where B is a symmetric matrix and A is a skew-symmetric matrix. Since matrices of this form also constitute a vector space $Sp(n;\square) \cap O(2n)$ is a Lie group of dimension m^2. (viii) The intersection $Sp(n;\square) \cap \bigcup(2n)$ is called a unitary sympletic group and denoted by $Sp(m)$.

It is a Lie group of dimension $2m+1$.

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9. Lemma: A topological group M is connected if it contains a connected subgroup N with a connected factor M/N

Proof: The natural mapping $\theta: M \to M/N$ is open i.e. turns open sets into open sets. In fact if $V \subset M$ then by the definition of factor topology a set $\theta(v) \subset M/N$ is open iff so is $\theta^{-1}(\theta(v)) \subset M$. But it is clear that the latter is the union $\bigcup_{y \in v} yN$ of all cosets yN, $y \in V$ and hence coincides with the union $\bigcup_{n \in N} Vn$ of all shifts of V by the elements $n \in N$. So if V and hence any V_n is open then the set $\theta^{-1}(\theta(v))$ and hence $\theta(v)$ are open. Now let $M = V \bigcup W$, whereare non-empty sets, then $M/N = \theta(v) \bigcup \theta(w)$, where $\theta(v)$ and $\theta(w)$ are also nonempty and open. So $\theta(v) \bigcup \theta(w) \neq \phi$ is nonempty either (since the space M/N is assumed to be connected). Let $\theta(b) \in \theta(v) \bigcap \theta(w)$, the inclusion $\theta(b) \in \theta(v)$ implies that the coset $\theta(b) = bN$ intersects $\bigcup$ and the inclusion $\theta(b) \in \theta(N)$ implies that the coset intersects W. One can have $bN = v_1 \bigcap w_1$ where $v_1 = bN \bigcap V$ and $w_1 = bN \bigcap W$ are open in bN. Since bN (together with N) is connected. This is possible iff $v_1 \bigcap w_1 \neq \phi$ and hence $v \bigcap w \neq \phi$. Consequently M is connected.

Conclusion:

In the above research article four important propositions are presented with elegent proofs. The concepts namely matrix Lie groups admitting the Cayley construction. A generalization of the Cayleyconstruction and the groups possessing $\ln -$ images are briefly discussed. In the context of future research one can derive the formulae for (i) the values of smooth functions in the normal neighbourhood of the identity of a Lie group and (ii) values of smooth functions on the product of the elements.In addition to the above research discussion the necessary and sufficient condition for a topological group to be Hausdorff is derived. In addition to this the necessary condition for a topological group to be connected has been proved. In the context of future research one can extend these ideas to prove that a Lie group is always parallelizable.

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