TEST Engineering & Management

March-April 2020
ISSN: 0193-4120 Page No. 5872 - 5877

Some Special Characteristics of Atoms in Lattice Ordered Loops

R. Sunil Kumar [1], V. B .V .N. Prasad [2], B.Rama Devi [3]
[1, 2] *Koneru Lakshmaiah Education Foundation*, *Vaddeswaram*, Guntur (Dt), A.P., India.
[3] Lakireddy Bali Reddy College of Engineering, Mylavaram, A.P., India.

Article Info
Volume 83
Page Number: 5872 - 5877
Publication Issue:
March - April 2020

Article History
Article Received*: 24 July 2019*
Revised*: 12 September 2019*
Accepted*: 15 February 2020*
Publication*: 30 March 2020*

Abstract:

In this manuscript, we consider that L is a lattice ordered loop. Further, we discuss some important characteristics of atoms in lattice ordered loops. We initiate the concepts of positive and negative atoms, dual atom, atomic lattice, meet irreducible element, join irreducible element, descending chain condition and ascending chain condition, right Archimedean property. Here there are two topics, one is about atoms in lattice ordered loops and the other is about Archimedean property.

Keywords: *Loop,partial ordered loop, lattice ordered loop, lattices etc.*

I. Introduction

In 1942 Garrett Birkhoff [1] was originallyoriginated the concept of lattice ordered groups. In 1964 and 1967 Garrett Birkhoff [1] was established various crucial properties of lattice ordered groups. In 1970 Trevor Evans [2] described about lattice ordered loops and quasi groups. In 1971 "Richard Hubert Bruck" was made a survey of binary systems. In 1990 Hala made description on quasi groups and loops. In 2013, V. B. V.N. Prasad & J V Rao, were published Characterization of Quasi groups and Loops [3], Classification of partially ordered loops and lattice ordered loops [4], in 2014 they were published Classification of "Normal sub loop and Ideal of loops" [5] and in 2014 Cones in Lattice ordered loops [6] were published in lattice ordered loops.

II ATOMS IN LATTICE ORDERED LOOPS:

Definition2.1: A system $(S,+, \leq)$, where S is set with non-zero cardinality,+ is a binary operation on S , & $\leq$ is a "binary relation on S "satisfying

(i) "(S, +) is a loop"

(ii) "$(S, \leq)$ is a partially ordered set"

(iii) "The translations $x \longmapsto a + x$ and $x \longmapsto x + b$ are ordered automorphisms of S,is called a partially ordered loop (briefly P.O. loop)".

Example 2.1: $(Z, +,\leq)$ is a (P.O) partially ordered loop.

Definition 2.2: If in a "partially ordered loop, in which the partial order is a lattice order, then that loop is called as a lattice ordered loop".

Definition2.3: In a lattice ordered loop "L an element $x \in L$ is called positive element if $x \geq 0$".

Definition 2.4: In a lattice ordered loop L, a positive element which covers 0 is called an atom.

March-April 2020
ISSN: 0193-4120 Page No. 5872 - 5877

Note 2.1: An atom is never minimal

Note 2.2: An atom is an element that is minimal element among the nonzero elements. **Proof:** By definition of atom and by Note (2.1), an atom is an element that is minimal element among the nonzero elements.

Definition 2.5: In a lattice ordered loop L has a lower bound 0, andx ∈ L is an atom if "0 < x "there is no element "y ∈L ∋ 0 < y < x".

Examples: 2.2 In the power set P(X) of a nonempty set X, for any a∈ X, {a} is an atom. Because in (P(X),⊆) is a poset in a lattice ordered loop L with $\emptyset\ as$ minimal elementand hence all singleton sub sets {a} of P(X) are atoms in L.

Example 2.3: Let N be the set of all natural numbers. If (N, ≤) is a partly ordered set in a lattice ordered loop L, then 2 is the only atom because it is covered by the minimal element 1 in N and hence by def of atom 2 is the only atom in N by the usual relation ≤.

Example 2.4: If (N, |) is a partly ordered set in a lattice ordered loop L then every prime number is an atom.

Solution: If (N,|) is a partly ordered set in a lattice ordered loop L. Here "|" is a relation called "divides". So N is the set of natural numbers with the relation "divides". As prime has only divisors 1 and itself, so 1|p, for all primes p in N. Here 1 is covered by all primes in N. So every prime is an atom in (N,|).

Definition2.6: In a lattice ordered loop L, dually an element 'm' of a lattice bounded above is said to be a dual atom if m ≺ 1. That is m is covered by 1 or 1 covers m.

Example 2.5: In P(X), for any a∈ X, X - {a} is a dual atom.

Definition 2.7: A lattice ordered loop L bounded below is said to be an atomic lattice if for every a≠0 of L ∃ an atom p of L ∋ p ≤ a.

Definition 2.8: In a lattice ordered loop L, if for every descending sequence $a_1 \geq a_2 \geq \ldots\ldots$, of elements of L,$\exists n \in N$ ∋ $a_n = a_{n+1} = a_{n+2} = \ldots\ldots$then "L is said to satisfy the minimum condition or descending chain condition".

Definition 2.9: In a lattice ordered loop L, if for every ascending sequence $a_1 \leq a_2 \leq \ldots\ldots$ ..., of elements of L, $\exists n \in N$ ∋$a_n = a_{n+1} = a_{n+2} = \ldots\ldots$then L is said to fulfil the extremestate or ascending chain state.

Example 2.6: If N is the set of natural numbers and ≤ is the usual ordering and(N, ≤) is a partly ordered set in a lattice ordered loop L then it satisfies the minimum condition but does not satisfy the maximum condition.

Example 2.7: In the set of real numbers R, ≤ is the usual ordering and (R, ≤) is a partly ordered set in a lattice ordered loop L then it satisfies neither maximum nor minimum conditions.

Example 2.8: If X is an infinite set then its power set P(X) satisfies neither the maximum nor the minimum condition.

Theorem 2.1: In a lattice ordered loop L, which is bounded below and it satisfies the minimum condition, then it is atomic.

Proof: Let us assume that L be a lattice ordered loop and bounded below satisfying the minimum condition. Let o ≠ x ∈ L. Consider A = {a∈L / 0 < a ≤ x}.Since L satisfies the minimum condition, A has a minimal element say p.Then p is an atom and p ≤ x. Therefore ∀ $0 \neq x \in$ L, ∃an atom p ∋ p ≤x.Hence L is an atomic lattice.

Theorem 2.2: If in a lattice ordered loop L and an element p of L is an atom iff for each element x of L, either p ≤x or x ∧ p = 0.

Proof: Let us assume that L is a lattice ordered loop and p ∈ L is an atom.

Let x ∈ L. Then we have 0 ≤ x ∧ p ≤ p.

⇒Either x ∧ p = p or 0= x ∧ p, because p is an atom.

This implies either p ≤ x or x ∧ p = 0.

Conversely assume that for each element x of L.

March-April 2020
ISSN: 0193-4120 Page No. 5872 - 5877

Let $0 \leq x \leq p$.

Then by our assumption $x \wedge p = 0$ or $x = p$.

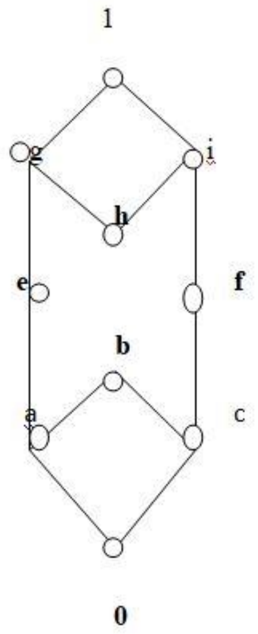
Then *either* $x = 0$(since $x \wedge p = x$) or $x = p$.

Therefore P is an atom.

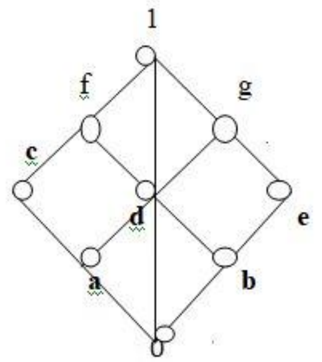
Definition 2.10: In a lattice ordered loop L,an element 'a' of Lis said to be meet irreducible if $a = a_1 \wedge a_2$ ($a_1, a_2 \in L$) then either $a = a_1$ or $a = a_2$. That is it cannot be decomposed by elements greater than 'a'.

Definition 2.11: In a lattice ordered loop L,an element 'a' of L is said to be join irreducible if $a = a_1 \vee a_2$ ($a_1, a_2 \in L$) then either $a = a_1$ or $a = a_2$. That is it cannot be decomposed by elements less than a.

Example 2.9: Find join and meet irreducible elements of the lattice ordered loop given below.



L2



L1

Solution: Join irreducible elements of $L_1 = \{0, a, b, c, e\}$

Meet irreducible elements of $L_1 = \{c, e, f, g, 1\}$

Join irreducible elements of $L_2 = \{0, a, c, e, f, h\}$

Meet irreducible elements of $L_2 = \{b, e, f, g, i, 1\}$

Note 2.3: Clearly the least element and every atom of a lattice bounded below is join irreducible while the greatest element and every dual atom of a lattice bounded above is meet irreducible by the above definitions (2.10), (2.11).

Note 2.4: In a chain, every element is meet irreducible as well as join irreducible.

Result 2.1: In a lattice ordered loop L, L is a chain iff every one of its elements is meet irreducible.

Proof: Suppose L is a chain and $a \in L$.

If $a = b \wedge c$ then either $a = b$ or c.

March-April 2020
ISSN: 0193-4120 Page No. 5872 - 5877

as b and c are comparable, $b \wedge c = b$ or c.

Hence a is meet irreducible.

Conversely suppose that every element of L is meet irreducible and let $a, b \in L$.

Now, by assumption $a \wedge b$ is meet irreducible and $a \wedge b = a \wedge b$.

This implies, $a \wedge b = a$ or $a \wedge b = b$.

Hence a and b are comparable.

Thus L is a chain.

Theorem 2.3: Ina lattice ordered loop L, every atom is join irreducible.

Proof: Let us assume that"L is a lattice ordered loop (L.O.L)" and $p \in L$ is an atom.

Claim: p is join irreducible.

Let $a_1, a_2 \in L \ni p = a_1 \vee a_2$

Implies $a_1 \leq p$ and $a_2 \leq p$

Implies "$0 \leq a_1 \leq p$ and $0 \leq a_2 \leq p$".

This implies, "$0 = a_1$ or $a_1 = p$ and $0 = a_2$ or $a_2 = p$" (since p is atom).

This leads to $p = 0 \vee a_2$ or $a_1 = p$ and $p = a_1 \vee 0$ or $a_2 = p$.

Then "$p = a_2$ or $a_1 = p$ and $p = a_1$ or $a_2 = p$".

Thus we have $a_1, a_2 \in L$ with $p = a_1 \vee a_2$.

Hence p equals to a_1 or p equals to a_2.

Therefore p is join irreducible

Hence every atom in a lattice ordered loop L is join irreducible.

Theorem 2.4: In a lattice ordered loop L, every dual atom is meet irreducible.

Proof: Let be a lattice ordered loop and $p \in L$ is dual atom in L.

Claim: p is meet irreducible.

Let $a_1, a_2 \in L$ such that $p = a_1 \wedge a_2$.

Then $p \leq a_1$ and $p \leq a_2$.

This implies $p \leq a_1 \leq 1$ and $p \leq a_2 \leq 1$.

So $p = a_1$ or $a_1 = 1$ and $p = a_2$ or $a_2 = 1$.

Then "$p = a_1$ or $p = 1 \wedge a_2$ and $p = a_2$ or $p = a_1 \wedge 1$".

This leads to "$p = a_1$ or $p = a_2$ and $p = a_2$ or $p = a_1$".

Thus we have $a_1, a_2 \in L$ with $p = a_1 \wedge a_2$.

Hence, $p = a_1$ or $p = a_2$.

Therefore p is meet irreducible.

Hence in a lattice ordered loop L, every dual atom is meet irreducible.

Definition 2.12: If L is a lattice ordered loop with descending chain condition (d.c.c.) on its positive elements, then these elements are called as positive atoms.

Note 2.5: In a lattice ordered loop L, any two positive atoms are orthogonal. That is, $x \wedge y = 0$, for all x, y in L.

Theorem 2.5: In a lattice ordered loop L, if a, b and c arepositive (+ve) atoms such that $a \wedge b = 0$ and $a \wedge c = 0$ then $a \wedge (b + c) = 0$.

Note 2.6: Similarly we can also verify that in a l.o. loop L, if a, b, c are positive (+ve) atoms such that $a \vee b = 0$ and $a \vee c = 0$ then $a \vee (b + c) = 0$

Proof: By taking the dual of the theorem.

Theorem 2.6: In a Lattice ordered loop L, if a is a positive (+ve) atom and $a + a > x > 0$, then $x = a$.

Theorem 2.7: In a Lattice ordered loop L,"a, b are positive (+ve) atoms and $a + b > x > 0$, then $x = a$ or $x = b$".

Theorem 2.8: In a Lattice ordered loop L, if a is a positive atom, $0 - a = -a + 0$. (i.e the inverses of left and right are equal).

March-April 2020
ISSN: 0193-4120 Page No. 5872 - 5877

Theorem 2.9: In a Lattice ordered loop L, if a, b are positive (+ve) atoms in L and a > x >$\bar{b}$, then x = 0 or x = a + $\bar{b}$.

We recall the following (without proof)

Theorem 2.10: (Birkhoff): "A lattice (L, ∨, ∧) is distributive if and only if, for any a, x, y, ∈ L, a ∧x = a ∧ y and a ∨ x = a ∨ y ⟹ x = y". (This property can be called as cancellation rule).

Lemma 2.1: In a lattice ordered loop L,-[a - (b∧ c)] + (a∨ b) = bfor all a, b, c∈L

Proof: Let L be a lattice ordered loop and a, b, c ∈ L.

"Now [a-(a∧ b)] + b = [(a-a) ∨ (a-b)] + b = [0 ∨ (a-b)] + b

= (0+b) ∨ [(a-b) + b] = b∨ a".

Hence b = *-[a-(a∧ b)] + (a∨b), for all a, b, c* ∈ L.

Lemma 2.2: Any lattice ordered loop is a distributive lattice.

Proof: Let L be a lattice ordered loop.

Let a ∨x = a∨y and a ∧ x = a ∧ y in a lattice ordered loop L.

Now y = - [a- (a ∧ y)] + (a ∨ y) (by lemma 2.1)

= - [a - (a ∧ x)] + (a ∨ x) = x (by lemma 2.1)

So by the Birkhoff theorem (2.10), L is a distributive lattice.

Hence any lattice ordered loop is a distributive lattice.

Definition 2.13: An element a∈ A is called a right Archimedean of A, if a ≥0 and if, for every element b, there is a positive integer n such that na≥ b.

Note 2.7: If all positive elements of A are right Archimedean, it is called to be right Archimedean.

Theorem 2.11: In a lattice ordered loop L, L is Archimedean, iffna≤b, n = 1, 2, …implies a ≤ 0.

In other words, in an ordered loop L, the concepts of the integrally closed and Archimedean are equivalent.

Proof: Let L be a lattice ordered loop and Archimedean.

If na≤ b, n = 1, 2, …. , and a > 0, then n_0a ≥ b for some n_0, which is a contradiction to na≤ b.

By definition of ordered loop, a must be non-positive.

That is a ≤0.

Conversely suppose if ordered loop is integrally closed and let 'a' be a positive element.

Suppose na≤ b (n = 1, 2…), then element a must be non-negative, which contradicts a > 0.

Hence for some natural number n_0, n_0a >b.

Hence L is Archimedean.

Theorem 2.12: In a lattice ordered loop L, L has an atom and if its atom is Archimedean, then it is a cyclic group.

Proof: Given that L is a lattice ordered loop.

Let p be atom of L, then we have the identities:

p + np = np + p (n = 1, 2…) and

p + (a + b) = (p + a) + b = (a + b) + p, for a, b of L⟶ (1)

In terminology of the theory of loops, p is in the center of L
since p is atom, there exists no element c such that a + b < c < p + (a + b),

a + b < c < (p + a) + b and a + b < c < (a + b) + p.

This shows the equality (1).

Thus the set {np / n = 0, ±1, 2…} is a cyclic group.

As p is Archimedean, there is aoptimistic integer such that 0 < a< np for any fixed optimistic element a.

Let n be such a minimal optimistic integer.

Then we have $(n-1)p \leq a < np$.

Thus $0 \leq a - (n-1)p < p$.

Since p is atom, $a = (n-1)p$.

Therefore, every element a of L is written in the form $a = np$ ($n = 0, \pm 1, 2...$)

By the above theorem (2.11), L must be cyclic group.

Corollary2.1: In an Archimedean ordered loop L. If L has an atom, it is isomorphic to the set of all integers.

Proof: Suppose L is an Archimedean ordered loop.

If L has atom then by the above theorem every element $a \in L$ be able to be articulated in the formula $a=np$, where $n \in I$, and hence it is isomorphic to the set of integers.

Hence if Archimedean ordered loop has atom then it is isomorphic to the set of all integers.

III. CONCLUSION:

This study makes it possible the following results: A lattice ordered loop, if it is bounded below satisfies the minimum condition then it is atomic. An equivalent condition established for an atom in a lattice ordered loop. Every element of a chain is meet irreducible as well as join irreducible**.** Ina lattice ordered loop, every atom is join irreducible where as every dual atom is meet irreducible. Any lattice ordered loop is a distributive lattice. Established an equivalent condition for an ordered loop to be Archimedean. If an ordered loop has atom and if its atom is Archimedean, then it is a cyclic group.

REFERENCES

[1]. Garrett.Birkhoff, Lattice Theory, 3rd edition, American Mathematical society, colloquium Publications, pp.160-163, 1967.
[2]. Evans. T, Lattice ordered loops and quasigroups, Journal of Algebra, 16, pp.218-226, 1970.
[3]. V. B. V.N. Prasad, J V Rao,Characterization of Quasi groups and Loops,IJSIMR, Volume 1, Issue 2, 2013,pp.95-102, 2013.
[4]. V. B. V. N. Prasad, J V Rao, Classification of partially ordered loops and lattice ordered loops, IJMA, Volume 4, Issue 10, pp.78-84, 2013.
[5]. V. B. V. N. Prasad, J V Rao, Categorization of Normal sub loop and Ideal of loops, ARPN Journal of Engineering and Applied Sciences, Volume 9, Issue 7, 2014, pp.1076-1079, 2014.
[6]. V. B. V. N Prasad, J V Rao, Cones in Lattice ordered loops, International Journal of Mathematics and Computer Applications Research (IJMCAR), Volume 4, Issue 4, pp.67-74, 2014.