Oscillatory and Asymptotic Behavior of Certain Generalized Fourth order Quasilinear Difference Equations

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Abstract

The fourth-order quasi-linear generalized difference equation (FOQGDE) is given below

$$\Delta^2_{a(l)}\left(p(k)\left|\Delta^2_{a(l)}\right|^{\gamma-1}\Delta^2_{a(l)}u(k)\right)+q(k)\left|u((k+3l))\right|^{\beta-1}u((k+3l))=0$$

Where,

γ and β - positive constants, and

$p(k)$ and $q(k)$ - positive real-valued function

Here, Sufficient conditions are attained for oscillation of each solution, when $\sum_{k=k_0}^{\infty}\left(\frac{k}{p(k)}\right)^{\frac{1}{r}}<\infty$ and $\sum_{k=k_0}^{\infty}\left(\frac{k}{p^{\frac{i}{r}}(k)}\right)<\infty$.

***Key words:** Fourth order generalized difference equation; Non-oscillatio; Oscillation; Asymptotic behavior;*

1. INTRODUCTION

Jerzy Popenda, et.al [19] defined Δ_a as $\Delta_a u(k)=u(k+1)-au(k)$ whilst ascertaining the behavior of solutions belongs to a specific type of difference equation, $\Delta_a$swas ignored for a long period. In [16], the Δ_a was generalized to $\Delta_{a(l)}$ and was defined as $\Delta_{a(l)}u(k)=u(k+l)-au(k)$ for the real-valued function (RVF) $u(k)$sand $l\in(0,\infty)$ The solutions of generalized a-difference equations, specifically the generalized Clairauts a^{-1}difference equation, generalized Euler a^{-1}difference equation, and the generalized a^{-1}

Bernoulli polynomial were attained. A solution $B_{a(n)}(k,l)$,was attained for the a^{-1}difference equation $u(k+1)-au(k)=nk^{n-1}$,for $n\in N(1)$

In the past years, several papers researched the oscillatory behaviors of solution of a neutral delay difference equation.

These equations became the mathematical solution for certain real-time problems [5, 7-14, 15, 20-22].

In this work, a ***FOQGDE*** given below is considered,

$$\Delta^2_{a(l)}\left(p(k)\left|\Delta^2_{a(l)}u(k)\right|^{\gamma-1}\Delta^2_{a(l)}u(k)\right)+q(k)\left|u((k+3l))\right|^{\beta-1}u((k+3l))=0$$

for all $k\in N(k_0)=\{k_0,k_0=l,....\}$ (2)

where,

$\Delta_{a(l)}$ - a^{-1}Difference operator and is given as $\Delta_{a(l)}u(k)=u(k+l)-au(k)$,

k_0 - Non-negative integer

Y and β - Positive constants,

$p(k)$ and $q(k)$ - Positive real sequences

Solution of Eqn. (2) is a real sequence $u(k)$. If any '4' consecutive values of are given, then a solution $u(k)$ could be defined recursively. $u(k)$ If this non-trivial solution is eventually negative or eventually positive, it is viewed as non-oscillatory, else it is oscillatory.

Thandapani and Selvaraj [28] and Thandapani, Pandian, Dhanasekaran, and Graef [27] regarded the non-linear difference equation (Eqn. (2)) providing solutions with oscillatory as well as asymptotic behaviors. In [28], the Eqn. (2) was elucidated under below condition.

$$\sum_{k=k_0}^{\infty}\left(\frac{k}{p(k)}\right)^{\frac{1}{\gamma}} = \infty \text{ and } \sum_{k=k_0}^{\infty}\left(\frac{k}{p^{\frac{1}{\gamma}}(k)}\right) = \infty \qquad (3)$$

In [27], the asymptotic behaviors of non-oscillatory solutions of Eqn. (2) were examined under the condition of Eqn. (3). In [26], the Eqn. (2) was considered under the conditions of Eqn.(3), or $\sum_{k=k_0}^{\infty}\left(\frac{k}{p(k)}\right)^{\frac{1}{\gamma}} < \infty$ and $\sum_{k=k_0}^{\infty}\left(\frac{k}{p^{\frac{1}{\gamma}}(k)}\right) = \infty$ or Eqn. (6), and the condition for the availability of non-oscillatory solutions of Eqn. (2) was attained.

For $\gamma = 1$, Eqn. (2) becomes

$$\Delta^2(p(k)\Delta^2 u(k)) + q(k)\left|u(k+3l)\right|^{\beta-1} u(k+3l) = 0 \; (4)$$

Solutions of Eqn. (4) showed the oscillatory as well as asymptotic behaviors, and they were discussed by Yan and Liu [30], and Thandapani and Arockiasamy [29] respectively

$$\sum_{k=k_0}^{\infty}\left(\frac{k}{p(k)}\right)^{\frac{1}{\gamma}} = \infty \text{ or } \sum_{k=k_0}^{\infty}\left(\frac{k}{p^{\frac{1}{\gamma}}(k)}\right) < \infty \qquad (5)$$

Later, Graef and Thandapani [15] extended the results in [30] to the generalized equation given below

$$\Delta(a(k)\Delta(b(k)(c(k)\Delta u(k)) + q(k)\left|u(k)\right|^{\beta-1} u(k) = 0,$$

Here, the solutions of Eqn. (2) which have the oscillatory behavior are discussed under the conditions in Eqn. 6,

$$\sum_{k=k_0}^{\infty}\left(\frac{k}{p(k)}\right)^{\frac{1}{\gamma}} < \infty \text{ or } \sum_{k=k_0}^{\infty}\left(\frac{k}{p^{\frac{1}{\gamma}}(k)}\right) < \infty \qquad (6)$$

Section II describes the classification of solutions of Eqn. (4) with non-oscillatory behaviors, and Section III delineates the attained sufficient conditions of the oscillation of all solutions of Eqn. (2). Examples that illustrate the results are proffered in Section IV.

The monographs of Agarwal, O'Regan Bohner, and Grace [2], and the references, [3-11, 17-19, 23-25], are referred for understanding the related outcomes of the fourth-order difference equations having oscillatory behavior.

I. PROPOSED METHODOLOGY

Here, certain basic results concerning the classification of non-oscillatory solutions of equation Eqn. (2) are defined and proved with no loss in generality, it focuses on the positive solutions, because if $u(k)$ satisfies Eqn. (2), then ${}^{-1}u(k)$ is a positive solution. Also, the conditions are developed and proved for the oscillation of the solutions of Eqn. (2).

II. CLASSIFICATION OF NONOSCILLATORY SOLUTIONS

Lemma 1: *If the afore-said Eqn. (2) gives the eventually positive solution* $u(k)$ *, then one among the succeeding four cases satisfies all-sufficient large* k *:*

$$\Delta_{a(l)}u(k) > 0, \Delta^2_{a(l)}u(k) > 0,$$

$$\text{(I)}\ldots \Delta_{a(l)}(p(k)\left|\Delta^2_{a(l)}u(k)\right|^{\gamma-1}\Delta^2_{a(l)}u(k)) > 0;$$

$$\Delta_{a(l)}u(k) < 0, \Delta^2_{a(l)} > 0,$$

$$\text{(II)} \quad \Delta_{a(l)}\left(p(k)\left|\Delta^2_{a(l)}u(k)\right|^{\gamma-1}\Delta^2_{a(l)}u(k)\right) > 0;$$

$$\Delta_{a(l)}u(k) > 0, \Delta^2_{a(l)} < 0,$$

$$\text{(III)} \; \Delta_{a(l)}\left(p(k)\left|\Delta^2_{a(l)}u(k)\right|^{\gamma-1}\Delta^2_{a(l)}u(k)\right) > 0;$$

$$\Delta_{a(l)}u(k) > 0, \Delta^2_{a(l)} < 0,$$

$$\text{(IV)}\ \Delta_{a(l)}\left(p(k)\left|\Delta^2_{a(l)}u(k)\right|^{\gamma-1}\Delta^2_{a(l)}u(k)\right) < 0;$$

Proof: From Eqn. (2),

$\Delta_{a(l)}\left(p(k)\left|\Delta^2_{a(l)}u(k)\right|^{\gamma-1}\Delta^2_{a(l)}u(k)\right) < 0;$ for all large k . andsss $\Delta_{a(l)}\left(p(k)\left|\Delta^2_{a(l)}u(k)\right|^{\gamma-1}\Delta^2_{a(l)}u(k)\right)$, $\Delta^2_{a(l)}u(k)$ and $\Delta_{a(l)}u(k)$ are one-signed and eventually monotonic. Subsequently, the succeeding eight cases are considered:

$$\Delta_{a(l)}u(k) > 0, \Delta^2_{a(l)}u(k) > 0,$$

$$\text{(a)}\ldots\ \Delta_{a(l)}\left(p(k)\left|\Delta^2_{a(l)}u(k)\right|^{\gamma-1}\Delta^2_{a(l)}u(k)\right) > 0;$$

$$\Delta_{a(l)}u(k) < 0, \Delta^2_{a(l)}u(k) > 0,$$

$$\text{(b)...}\ \Delta_{a(l)}\left(p(k)\left|\Delta^2_{a(l)}u(k)\right|^{\gamma-1}\Delta^2_{a(l)}u(k)\right) > 0;$$

$$\Delta_{a(l)}u(k) > 0, \Delta^2_{a(l)}u(k) < 0,$$

$$\text{(c)}\quad \Delta_{a(l)}\left(p(k)\left|\Delta^2_{a(l)}u(k)\right|^{\gamma-1}\Delta^2_{a(l)}u(k)\right) > 0;$$

$$\Delta_{a(l)}u(k) < 0, \Delta^2_{a(l)}u(k) < 0,$$

$$\text{(d)...}\ \Delta_{a(l)}\left(p(k)\left|\Delta^2_{a(l)}u(k)\right|^{\gamma-1}\Delta^2_{a(l)}u(k)\right) > 0;$$

$$\Delta_{a(l)}u(k) > 0, \Delta^2_{a(l)}u(k) > 0,$$

$$\text{(e)}\ldots\ \Delta_{a(l)}\left(p(k)\left|\Delta^2_{a(l)}u(k)\right|^{\gamma-1}\Delta^2_{a(l)}u(k)\right) < 0;$$

$$\Delta_{a(l)}u(k) < 0, \Delta^2_{a(l)}u(k) > 0,$$

$$\text{(f)}\ldots\ \Delta_{a(l)}\left(p(k)\left|\Delta^2_{a(l)}u(k)\right|^{\gamma-1}\Delta^2_{a(l)}u(k)\right) < 0;$$

$$\Delta_{a(l)}u(k) < 0, \Delta^2_{a(l)}u(k) < 0,$$

$$\text{(g)}\ldots\ \Delta_{a(l)}\left(p(k)\left|\Delta^2_{a(l)}u(k)\right|^{\gamma-1}\Delta^2_{a(l)}u(k)\right) < 0;$$

$$\Delta_{a(l)}u(k) > 0, \Delta^2_{a(l)}u(k) < 0,$$

$$\text{(g)}\ldots\ \Delta_{a(l)}\left(p(k)\left|\Delta^2_{a(l)}u(k)\right|^{\gamma-1}\Delta^2_{a(l)}u(k)\right) < 0;$$

$$\Delta_{a(l)}u(k) < 0, \Delta^2_{a(l)}u(k) < 0,$$

$$\text{(h)}\ldots\ \Delta_{a(l)}\left(p(k)\left|\Delta^2_{a(l)}u(k)\right|^{\gamma-1}\Delta^2_{a(l)}u(k)\right) < 0;$$

If eventually $\Delta_{a(l)}u(k)$, $\Delta^2_{a(l)}u(k)$ and then. $\lim_{k\to\infty} u(k) = -\infty$ This contradicts the positivity of the considered solution ‘ $u(k)$ ’. Hence, the cases (d) and (h) could not hold. Similarly, for $\Delta_{a(l)}\left(p(k)\left|\Delta^2_{a(l)}u(k)\right|^{\gamma-1}\Delta^2_{a(l)}u(k)\right) < 0;$, if $\Delta_{a(l)}\left(p(k)\left|\Delta^2_{a(l)}u(k)\right|^{\gamma-1}\Delta^2_{a(l)}u(k)\right) < 0;$ then $\lim_{k=\infty}\left(p(k)\left|\Delta^2_{a(l)}u(k)\right|^{\gamma-1}\Delta^2_{a(l)}u(k)\right) = -\infty;$ that is, $\Delta^2_{a(l)}u(k) < 0$ for large k . This observation eliminates the cases (e) and (f) and hence, the lemma-1 is prosssved.”

Remark 2: *If* $a = \gamma = 1$ *,then Lemma Eqn. (1) reduces to Lemma 3 of [23].*

Lemma 3: *If Eqn. (2) of type (IV) gives a positive solution* $u(k)$ *, then there will be a positive number c in order that the succeeding inequality satisfies all large* k *:*

$$u(k+3l) \geq u(k) \geq ck\Delta_{a(k)}u(k) \tag{7}$$

Proof. Consider $u(k)$ as a positive solution of Eqn. (2) type (IV). If $\Delta_{a(l)}u(k)$ is positive and decreasing,

$$u(k) > u(k) - u(k) = \sum_{s=0}^{k-1} \Delta_{a(l)}u(K+sl) \geq (k-K)\Delta_{a(l)}u(k), k \geq K$$

Here, there is a constant $c > 0$ and has sufficient large k , in order that Eqn. (7) is satisfied and hence, the lemma-3 is proved.

Lemma 4*:* *Let* $y(k)$ *be an RVF defined for* $k \geq K \in N_l(k_0)$ *in order that”*

$$y(k) > 0, \Delta_{a(l)}y(k) > 0 and \Delta^2_{a(k)}y(k) < 0 \mathit{for} \quad all\, k \geq K$$
(8)

Here, constant n *and an integer* $k_1 \geq K$ *appears, in order that”*

$$y(k) \geq nk\Delta_{a(l)}u(k), \mathit{for} k \geq K_1 \tag{9}$$

Proof: As of Eqn. (8), we get

$$v(k) > v(k) - av(k) - (a-1)\sum_{r=1}^{k-1} v(k+rl)$$

$$= \sum_{s=0}^{k-1} \Delta_{a(l)} v(K+sl) \geq (k-K) l \Delta_{a(l)} v(k)$$

For which, (a) follows immediately.

Lemma 5: *Consider* $v(k), q(k)$ *as RVFs defined for* $k \geq K$ *and* $q(k) > 0$ *for* $k \geq K$, *in order that*

$$\sum_{r=0}^{\infty} \frac{1}{q(k+rl)} < \infty$$

If $v(k)$ *satisfies*

$v(k) > 0, \Delta_{a(k)} v(k) < 0 and \Delta_{a(l)} (q(k) \Delta_{a(l)} v(k)), \sqrt{b^2 - 4ac}$

sss

for $k \geq K$, *then*

$$v(k) \geq -q(k) \Delta_{a(l)} v(k) \sum_{S=1}^{\infty} \frac{1}{q(k+rl)}, \ for k \geq K \quad (10)$$

Proof: Arbitrarily fix $k \geq K \in N_l(k_0)$. Clearly, $q(s)\Delta_{a(l)} u(s) \leq q(s) \Delta_{a(l)} u(k)$ for $s > k$. Divide the last inequality with $q(s)$ and sum them as of k to t,

$$-au(k) < u(k) - au(k) - (a-1)\sum_{r=1}^{t} u(k+rl)$$

we get

$$\leq \Delta_{a(l)} u(k) \sum_{r=1} \frac{1}{q(s+rl)}$$

which is Eqn. (10) with the limit as $t \to \infty$.

Lemma 6: If $\Delta_{a(l)} v(k) < 0, \Delta^2_{a(l)} v(k) > 0$ $\Delta_{a(\ell)} v(k) < 0$, and $\Delta_{a(l)} (w(k) \Delta^2_{a(l)} v(k) > 0$, then constant n and an integer $k \in N_l(k_0)$ appears, in order that the subsequent inequalities hold for $k \geq K$.

$$v(k) \geq w(k) \Delta^2_{a(l)} v(k) \rho(k), \quad (11)$$

$$v(k) \geq n \Delta_{a(l)} (w(k) \Delta^2_{a(l)} v(k)) k \rho(k), \quad (12)$$

Proof: Consider the conditions hold $k \geq K_0 \in N(k_0)$ when $u(k) = \Delta_{a(l)} v(k)$ satisfies the hypothesis of lemma 5, With

$$-\Delta_{a(l)} v(k) \geq r(k) \Delta_{a(l)} v(k) \sum_{s=0}^{\infty} \frac{1}{w(k+sl)}, k \geq K_0$$

By summing the last inequality as of k to $t-1$ and utilizing the fact "$v(t) > 0$ and $w(k)\Delta^2_{a(l)} v(k)$ is increasing", we get

$v(k) \geq w(k) \Delta^2_{a(l)} v(k) \sum_{s=k}^{t-1} \left(\sum_{t=0}^{\infty} \frac{1}{(s+tl)} \right)$, which is Eqn. (11) as $t \to \infty$. When Lemma 4 is applied to $u(k) = w(k) \Delta^2_{a(l)} v(k)$, a constant $k > 0$ and an integer $K > K_0$ appears, in order that

$$w(k) \Delta^2_{a(l)} v(k) \geq nk \Delta_{a(l)} (w(k) \Delta^2_{a(l)} v(k)), k \geq K \quad (13)$$

By combining Eqn. (11) and Eqn. (13), Eqn. (12) is attained and hence, the lemma-6 is proved.

Lemma 7: *If Eqn. (2) of type (II) gives a positive solution* $u(k)$, *then a positive number* c *appears, in order that the subsequent inequalities hold for all large* k *:*

$$u(k+3l) \geq p(k)^{\frac{i}{r}} \psi(k+3l) \Delta_l^2 u(k), \quad (14)$$

$$(u(k+3l))^r \geq ck (\psi(k+3l))^r \Delta_{a(l)} (p(k) (\Delta^2_{a(l)} u(k))^r), \quad (15)$$

$$Where, \ \psi(k) = \sum_{s=k}^{\infty} \frac{(s-(k-1)l)}{p^{\frac{1}{r}}(s)}$$

Proof: The proof could be modeled as that of Lemma 4 (ii) of Thandapani and Arockiasamy [24], and therefore, the details are ignored.

Lemma 8: *Let* $Y \leq 1$. *If Eqn. (2) of type (IV) gives a positive solution* $u(k)$, *then a positive number* c appears, *in order that the subsequent inequality holds for all large* k *:*

$$u((k+3l)) \geq ck \left| \Delta_{a(l)} \left(p(k) \left| \Delta^2_{a(l)} u(k) \right|^{r-1} \Delta^2_{a(l)} u(k) \right) \right|^{\frac{1}{r}} \psi((k+3l)) \quad (16)$$

Proof: Since $\Delta_{a(l)} \left(p(k) \left| \Delta^2_{a(l)} u(k) \right|^{v-1} \Delta^2_{a(l)} u(k) \right)$ is decreasing, we find that

$$\Delta_{a(l)} \left(p(s) \left| \Delta^2_{a(l)} u(s) \right|^{r-1} \Delta^2_{a(l)} u(s) \right)$$

$$\leq \Delta_{a(l)} \left(p(k) \left| \Delta^2_{a(l)} u(k) \right|^{r-1} \Delta^2_{a(l)} u(k) \right), s \geq k$$

By summing the last inequality as of k to $s-1$, and utilizing the fact $\Delta^2_{a(l)} u(k) < 0$, we get

$$\Delta^2_{a(l)}u(s) \le -\left|\Delta_{a(l)}\left(p(k)\left|\Delta^2_{a(l)}u(k)\right|^{r-1}\Delta^2_{a(l)}u(k)\right)\right|^{\frac{1}{r}}$$

$$\times\left(\frac{s-(k-1)}{p(s)}\right)^{\frac{1}{r}}, s \ge k$$

As the $\lim\limits_{k\to\infty}\Delta_{a(l)}u(k)=\eta\ge 0$ is finite, and $\frac{1}{r}\ge 1$, by summing the last inequality as of k to ∞, we get

$$\Delta_{a(l)}u(k) \ge \left|\Delta_{a(l)}\left(p(k)\left|\Delta^2_{a(l)}u(k)\right|^{r-1}\Delta^2_{a(l)}u(k)\right)\right|^{r-1}$$

$$\times\sum_{s=k}^{\infty}\left(\frac{s-(k-1)l}{p(s)}\right)$$

$$=\left|\Delta_{a(l)}\left(p(k)\left|\Delta^2_{a(l)}u(k)\right|^{r-1}\Delta^2_{a(l)}u(k)\right)\right|^{\frac{1}{r}}$$

$$\ge\left|\Delta_{a(l)}\left(p(k)\left|\Delta^2_{a(l)}u(k)\right|^{r-1}\Delta^2_{a(l)}u(k)\right)\right|^{\frac{1}{r}}\psi(k+3l)$$

By combining the last inequality and the inequality of Eqn. (7), we get Eqn. (16). This proves the lemma 8.

Lemma 9: *Let $\beta<1<\gamma$ Then the condition*

$$\sum_{r=0}^{\infty}\left(\frac{1}{p(k_0+rl)}\sum_{s=0}^{k-1}(k-(k_0+sl))(k_0+sl)^{\beta}q(k_0+sl)\right)^{\frac{1}{Y}}=\infty \quad (17)$$

implies

$$\sum_{r=0}^{\infty}(k_0+rl)^{\frac{\rho}{r}}\psi^{\beta}(k_0+(r+1)l)q(k_0+rl)=\infty \quad (18)$$

Proof: If Eqn. (17) holds, then for any $K>k_0$, we get

$$\sum_{r=0}^{\infty}\left(\frac{1}{p(k+rl)}\right)^{-\frac{1}{r}}\left(\sum_{s=0}^{k-1}(k-(K+sl))(K+sl)^{\beta}q(k+sl)\right)^{\frac{1}{r}}=\infty \quad (19)$$

and choose $K>1$, in order that $\psi(k)\le 1$ for $k\ge K$. As of Eqn. (6), the Eqn. (19) implies that

$$\lim_{k\to\infty}\sum_{s=0}^{k-1}(k-(K+sl))(K+sl)^{\beta}q(K+sl)=\infty$$

Since k is increasing and $\lim\limits_{k\to\infty}=\infty$, by Stolz's theorem [6], we get

$$\lim_{k\to\infty}\frac{\sum\limits_{s=0}^{k-1}(k-(k+sl))(k+sl)^{\beta}q(k+sl)}{k}$$

$$\frac{\Delta_{a(l)}\left(\sum\limits_{s=0}^{k-1}(k-(K+sl))(k+sl)^{\beta}q(K+sl)\right)}{(\Delta_{a(l)}k)}$$

$$=\lim_{k\to\infty}\sum_{s=0}^{k}(k+sl)^{\beta}q(K+sl)\in(0,1]$$

Hence, a constant $d>0$s, and an integer $K_1>K$ appears, in order that

$$\sum_{S=0}(k-(K+sl))(K+sl)^{\beta}q(K+sl)\ge dk, \text{ for}$$

every

$$k\ge K_1. \quad (20)$$

For all $k\ge K_1$, utilizing summation by parts, we get

$$\sum_{s=0}^{k-1}p(K_1+sl)^{-\frac{1}{r}}\left(\sum_{t=0}^{s-1}(s-(K+tl))(K+tl)^{\beta}q(K+tl)\right)^{\frac{1}{r}}$$

$$=\sum_{s=0}^{k-1}\Delta^2_{a(l)}\psi(K_1+sl)\left(\sum_{t=0}^{s-1}(s-(k+tl))(K+tl)^{\beta}q(K+tl)\right)^{\frac{1}{r}}$$

$$=\left[\Delta_{a(l)}\psi(K_1+sl)\left(\sum_{t=0}^{s-1}(s-(K+tl))(K+tl)^{\beta}q(K+tl)\right)^{\frac{i}{r}}\right]_0^k \quad (21)$$

$$-\sum_{s=0}^{k-1}\Delta_{a(l)}\psi(K_1+(s+1)l)\Delta_{a(l)}$$

$$\times\left(\sum_{t=0}^{s-1}s-(k+tl))(K+tl)^{\beta}q(K+tl)\right)^{\frac{1}{r}}$$

Using Mean value theorem, we get

$$\Delta_{a(l)}\left(\sum_{t=0}^{s-1}(s-(K+tl))(K+tl)^{\beta}q(K+tl)\right)^{\frac{1}{r}}$$

$$\le\frac{1}{\gamma}\left(\sum_{t=0}^{s}(s-(K+tl))(K+tl)^{\beta}q(K+tl)\right)^{\frac{1}{r}-1} \quad (22)$$

$$\times\sum_{t=0}^{s}(K+tl)^{\beta}q(K+tl).$$

As of the inequalities in Eqn. (21) and Eqn. (22) and using Eqn. (20), and fact "$\frac{1}{\gamma} \le 1$as well as $\Delta_{a(l)}\psi(k)$ is a negative function", we get

$$p(K_1+sl)^{-\frac{i}{r}}\left(\sum_{t=0}^{s-1}(s-(K+tl))(K+tl)^{\beta}q(K+tl)\right)^{\frac{1}{r}}$$

$$\sum_{s=0}^{k-1} \le d_1 - d_2\sum_{s=0}^{k-1}\Delta_{a(l)}\psi(K_1-(s+1)l)(k_1+sl)^{\frac{1}{r}-1} \tag{23}$$

$$\times\sum_{t=0}^{s}(K+tl)^{\beta}q(K+tl), k \ge K_1;$$

where

$$d_1 = -\Delta_{a(l)}\psi(K_1)\left(\sum_{t=0}^{K_1-1}(K_1-(K+tl))(K+tl)^{\beta}q(K+tl)\right)^{\frac{1}{r}} > 0$$

$$d_2 = \frac{1}{\gamma}d^{\frac{1}{\gamma}-1} > 0$$

Now, by implementing the fact

$$0 \le \frac{(1-\gamma)(\beta-1)}{\gamma} = \frac{\beta}{\gamma} - \left(\frac{1}{\gamma} - 1 + \beta\right)$$

the Eqn. (23) becomes

$$\sum_{S=0}^{k-1}p(K_1+sl)^{-\frac{1}{\gamma}}\left(\sum_{t=0}^{s-1}(s-(K+tl))(K+tl)^{\beta}q(K+tl)\right)^{\frac{1}{\gamma}}$$

$$\le d_1 - d_2\sum_{s=0}^{k-1}\Delta_{a(l)}\psi(K_1+(s+1)l)\left(\sum_{t=0}^{s}(K+tl)^{\frac{\beta}{\gamma}}q(K+tl)\right)$$

$$\le d_3 + d_2\sum_{s=0}^{k-1}\psi(K_1+(s+1)l)(K_1+sl)^{\frac{\beta}{\gamma}}q(K_1+sl)$$

$$\le d_3 + d_2\sum_{s=0}^{k-1}(K_1+sl)^{\frac{\beta}{\gamma}}\psi(K_1+(s+1)l)q(K_1+sl),$$

$k \ge K_1$,

Where,

$$d_3 = d_1 + d_2\psi(K_1+l)\sum_{t=0}^{K_1-1}(K_1+tl)^{\frac{\beta}{\gamma}}q(K_1+sl)$$

Let $K \to \infty$, we infers that Eqn. (17) implies Eqn. (18). Therefore, it proves the provided Lemma 9.

Certain useful propositions that have an imperative role in proving the main outcomes are defined and proved in the succeeding Section (3).

Proposition 10. *Let $\beta > \gamma$ If the Eqn. (2) in type (II) has a positive solution $u(k)$, then*

$$\sum_{k=0}^{\infty}(k_0+rl)q(k_0+rl)\psi^{\beta}(k_0+(r+3)l) < \infty \tag{24}$$

Proof: If Eqn. (2) of type (II) has a positive solution of $u(k)$, then, $u(k) \approx c\,\psi(k)$ as $k \to \infty (0 < c < \infty)$. Else if $u(k)$ is of type (I), (III), or (IV) then $\lim_{k\to\infty} u(k) = c_0 \in (0, \infty]$.Furthermore, $\lim_{k\to\infty}\psi(k) = 0$.Therefore, $\lim_{k\to\infty}\frac{u(k)}{\psi(k)} = \infty$, is a contradiction. There exists an integer $K \ge k_0$ in order that

$$d_4\psi(k) \le u(k) \le 2d_4\psi(k),\ for\, all\, k \ge K \tag{25}$$

Now, the Eqn. (2) is multiplied with k, and they are totaled as of K to $k-1$. Therefore, we get

$$k\Delta_{a(l)}\left(p(k)\left(\Delta^2_{a(l)}u(k)\right)^{\gamma}\right)$$

$$-al\ p(k+l)\left(\Delta^2_{a(l)}u(k+l)\right)^{\gamma} + (1-a)\sum_{\gamma=0}^{k}(K+\gamma l)\Delta_{a(l)}$$

$$\times\left(p(K+rl)\left|\Delta^2_{a(l)}u(k+l)\right|\right)^{r} + (1-a)\sum_{r=0}^{k}(K+rl)\Delta_{a(l)}$$

$$+\sum_{r=0}^{k}(r+a(1+r)l)$$

$$\times\left(p(K+rl)\left|\Delta^2_{a(l)}u(K+rl)\right|^{r-1}\Delta^2_{a(l)}u(K+rl)\right)$$

$$+\sum_{s=0}^{k-1}(K+sl)q(K+sl)u^{\beta}(K+(s+3)l) = c$$

(26)

Here, c is a constant. As of Eqn. (14) and Eqn. (25), we get

$$p(k)\left(\Delta^2_{a(l)}u(k)\right)^{\gamma} \le 2^{\gamma}d_4^{\gamma}\ for\, k \ge K \tag{27}$$

From Eqn. (26) and Eqn. (27) and with the fact "$\Delta_{a(l)}\left(p(k)(\Delta^2_{a(l)}u(k))^{\gamma}\right) > 0$", we get

$$d_4^{v}\sum_{k=0}^{\infty}(k_0+kl)q(k_0+kl)\psi^{\beta}\left(k_0+(k+3l)\right) < \infty$$

this gives Eqn. (24). Therefore, the proof is attained.

Proposition 11. Consider $\beta < \gamma$ *If the Eqn. (2) of type (II) has a positive solution* $u(k)$*, then*

$$\sum_{k=0}^{\infty} (k_0 + kl)^{\frac{\beta}{\chi}} q(k_0 + kl)\psi^{\beta}(k_0 + (k+3)l) < \infty \quad (28)$$

Proof. For a positive solution $u(k)$ of Eqn. (2), consider

$K \geq k_0$ in order that type (II) and Eqn. (15) hold for all $k \geq K$, which is expressed by

$$A(k) = \Delta_{a(l)}(p(k)(\Delta^2_{a(l)}u(k))^{\gamma})$$

Then, as of the Eqn. (2) and by implementing Mean value theorem, we get

$$\Delta_{a(l)}\left(A(k)^{\left[1-\frac{\beta}{\gamma}\right]} \right) = \left(1 - \frac{\beta}{\gamma}\right) t^{-\frac{\beta}{\gamma}} \Delta_{a(l)}(A(k))'$$

$$A((k+1)) < t < A(k),$$

$$\leq -\frac{(\gamma - \beta}{\gamma} q(k)(u(k+3l))^{\beta} A^{-\frac{\beta}{\gamma}}(k)$$

$$\leq -\frac{(\gamma - \beta)}{\gamma} c^{\frac{\beta}{\gamma}} k^{\frac{\beta}{\gamma}} \psi^{\beta}(k+3l)q(k)$$

Here, the inequality Eqn. (15) is utilized. By Summing the last inequality as of K to $k-1$, we get

$$\frac{(\gamma - \beta)}{\gamma} c^{\frac{\beta}{\gamma}} \sum_{s=0}^{k-1} (K + sl)^{\frac{\beta}{\gamma}} q(K + sl)\psi^{\beta}(K + (s+3)l)$$

$$< A^{1-\frac{\beta}{\gamma}}(k) < \infty,$$

this results in Eqn. (28). Therefore, the proof ia attained.

Proposition 12. *Let* $\gamma \leq 1 \leq \beta$ *If the Eqn. (2) of type (IV) has a positive solution* $u(k)$*, then*

$$\sum_{k=0}^{\infty} (k_0 + kl) q(k_0 + kl)\psi^{\beta}(k_0 + (k+3)l) < \infty \quad (29)$$

Proof. For a positive solution $u(k)$ of Eqn. (2), choose an integer $K \geq k_0$ in order that type (IV) and Eqn. (16) hold for all $k \geq K$, which is expressed as,

$A(k) = \Delta_{a(l)}(p(k)\left|\Delta^2_{a(l)}u(k)\right|^{\gamma-1}\Delta^2_{a(l)}u(k))$. Then, as of the Eqn. (2), and by implementing Mean value theorem, the subsequent equations are attained,

$$-\Delta_{a(l)}(|A(k)|^{1-\frac{\beta}{\gamma}}) \geq -\frac{(\gamma - \beta)}{\gamma} |A(k)|^{-\frac{\beta}{\gamma}} \left(-\Delta_{a(l)}A(k)\right)$$

$$= -\frac{\gamma - \beta}{\gamma} |A(k)|^{1-\frac{\beta}{\gamma}} q(k)u^{\beta}(k+3l)$$

From the Eqn. (16) and last inequality, we could see $c_2 > 0$ in order that

$$-\Delta_{a(l)}\left(|A(k)|^{1-\frac{\beta}{\gamma}} \right)$$

$$\geq \frac{(\beta - \gamma)}{\gamma} |A(k)|^{-\frac{\beta}{\gamma}} c_2^{\beta} k^{\beta} q(k) |A(k)|^{\frac{\beta}{\gamma}} \psi^{\beta}(k+3l), k \geq K.$$

By implementing the fact $\beta - \gamma$, that is, $k^{\beta} \geq k$ for all

$k \geq \max\{1, K\} = K_1$, we get

$$-\Delta_{a(l)}\left(|A(k)|^{1-\frac{\beta}{\gamma}} \right) \geq \frac{(\beta - \gamma)}{\gamma} c_2^{\beta} k q(k)\psi^{\beta}(k+3l), k \geq K_1$$

By summing the last inequality as of K_1 to $k-1$, we get

$$-|A(k)|^{1-\frac{\beta}{\gamma}} + |A(K_1)|^{1-\frac{\beta}{\gamma}}$$

$$\geq \frac{(\beta - \gamma)}{\gamma} C_2^{\beta} \sum_{S=0}^{K-1} (K_1 + sl)q)K_1 + sl)\psi^{\beta}(K_0 + (s+3)l)$$

this results in Eqn. (29). Therefore, the proof is attained.

Proposition 13. *Let* $\beta < 1 \leq \gamma$ *If the Eqn. (2) of type (IV) has a positive solution* $u(k)$*, then*

$$\sum_{k=0}^{\infty} p(k_0 + kl)^{-\frac{1}{\gamma}} \times \left(\sum_{s=0}^{k-1} (k - (k_0 + s)l)(k_0 + kl)^{\beta} q(k_0 + sl) \right)^{\frac{1}{\gamma}} < \infty, \quad (30)$$

Proof: Presume that the positive solution $u(k)$ of Eqn. (2), type (IV) holds for all $k \geq K \notin N(k_0)$ With Lemma 3, a constant

$c_1 > 0$, and an integer $K_1 \geq K$ appears, in order that Eqn. (7) holds for all $k \geq K_1$.By summing the Eqn. (2) as of K_1 to $k-1$, we get

$$-\Delta_{a(l)}(p(k)\left|\Delta^2_{a(l)}u(k)\right|^{\gamma-1}\Delta^2_{a(l)}u(k))$$
$$\geq \sum_{S=0}^{k-1} q(K_1+sl)u^{\beta}(K_1+(s+3)l), k \geq K_1$$

By totaling the last inequality as of K_1 to $k-1$, the resultant is,

$$-\Delta^2_{a(l)}u(k) \geq \frac{1}{p^{\frac{1}{\gamma}}(k)}$$
$$\times\left(\sum_{s=0}^{k-1}(k-(K_1+s)l)q(K_1+sl)u^{\beta}(K_1+(s+3)l\right)^{\frac{1}{\gamma}}, k \geq K_1$$

As of the last inequality, we figured out,

$$-\Delta_{a(l)}(\Delta_{a(l)}u(k))^{1-\beta} \geq (1-\beta)(\Delta_{a(l)}u(k))^{-\beta}(-\Delta^2_{a(l)}u(k))$$
$$\geq (1-\beta)(\Delta_{a(l)}u(k))^{-\beta}\frac{1}{p^{\frac{1}{\gamma}}(k)}$$
$$\times\left(\sum_{s=0}^{k-1}(k+(K_1-s)l)q(K_1+sl)u^{\beta}(K_1+(s+3)l)\right)^{\frac{1}{\gamma}}, k \geq K_1$$

Since $\Delta_{a(l)}u(k)$ is decreasing, that is,

$(\Delta_{a(l)}u(k))^{-\beta} \geq (\Delta_{a(l)}u(s))^{-\beta}$ for $k \geq s$, we get

$$-\Delta_{a(l)}\left((\Delta_{a(l)}u(k))^{1-\beta}\right) \geq \left(1-\beta\right)\frac{1}{p^{\frac{1}{\gamma}}(k)}$$
$$\left(\begin{array}{l}\sum_{s=0}^{k-1}(K_1+(k-s))q(K_1+sl) \\ (\Delta_{a(l)}u(K_1+sl))^{-\beta\gamma}u^{\beta}(K_1+(s+3)l\end{array}\right)^{\frac{1}{\gamma}}$$
$$=(1-\beta)\frac{1}{p^{\frac{1}{\gamma}}(k)}$$
$$\times\left(\begin{array}{l}\sum_{s=0}^{k-1}(K_1+(k-s)l)q(K_1+sl)((K_1+sl)\Delta_{a(l)}u(K_1+sl))^{-\beta\gamma} \\ \times\left((K_1+sl)\right)^{\beta\gamma}u^{\beta}(K_1+(s+3)l\right)\end{array}\right)^{\frac{1}{\gamma}}$$

for $k \geq K_1$.

Now, using Eqn. (7), we get

$$-\Delta_{a(l)}(\Delta_{a(l)}u(k))^{1-\phi} \geq (1-\beta)\frac{c_1^{\beta}}{p^{\frac{1}{\gamma}}(k)}$$
$$\left(\begin{array}{l}\sum_{s=0}^{k-1}K_1+(k-s)l)q(K_1+sl)u^{-\beta\gamma} \\ (K_1+sl)(K_1+sl)^{\beta\gamma}u^{\beta}(K_1+(k+3)l)\end{array}\right)^{\frac{1}{\gamma}}, k \geq K_1$$

Since $\Delta_{a(l)k}u(k+3l) \leq \Delta_{a(l)}u(K_1 l+j)k \geq K_1$, a constant $C > 0$, and an integer $K_2 \geq K_1$ appears, in order that $u(k+3l) \leq ck$ for $k \geq K_2$.

Therefore, the fact $\beta(1-\gamma) \leq 0$ implies

$u^{\beta(1-\gamma)}(k+3l) \geq c^{\beta(1-\gamma)}k^{\beta(1-\gamma)}$ for $k \geq K_2$ Hence, we get

$$-\Delta_{a(l)}(\Delta_{a(l)}u(k))^{1-\beta} \geq (1-\beta)c^{\frac{\beta(1-\gamma)}{\gamma}}\frac{c_1^{\beta}}{p^{\frac{1}{\gamma}}(k)}$$
$$\times\left(\begin{array}{l}\sum_{s=0}^{k-1}(K_2+(k-s)l)q(K_2+sl) \\ (K_2+sl)^{\beta(1-\gamma)}(K_2+sl)^{\beta\gamma}\end{array}\right)^{\frac{1}{\gamma}}, k \geq K_2$$

By totaling the last inequality as of K_2 to ∞, we attain

$$(\Delta_{a(l)}u(K))^{1-\beta} \geq N\sum_{K=0}^{\infty}\frac{1}{P^{\frac{1}{\gamma}}(k)}$$
$$\times\left(\sum_{S=0}^{K-1}(K_2+(k-s)l)q(K_2+sl)(K_2+sl)^{\beta}\right)^{\frac{1}{\gamma}}$$

Where $N=(1-\beta)c^{\frac{\beta(1-\gamma)}{\gamma}}c_1^{\beta} > 0$, which results in Eqn. (30), and therefore, the proof is attained.

III. OSCILLATION THEOREMS

Theorem 14. *Let* $\beta \geq 1 > \gamma$. *If*

$$\sum_{k=0}^{\infty}(k_0+kl)q(k_0+kl)\psi^{\beta}((k+3)l) = \infty, \qquad (31)$$

then each solution of Eqn. (2) is oscillatory.

Proof: Consider, to contrary, If Eqn. (2) gives a positive solution $u(k)$, then $u(k)$ falls under one of the '4' types (I to IV) previously mentioned in Lemma 1. Hence, it is almost enough to confirm that each case led to a contradiction of Eqn. (31).

Case (I): The Eqn. (2) of type (I) gives a positive solution for all $k \geq K$. Since $\Delta_{a(l)}u(k) \geq \Delta_{a(l)}u(k), k \geq K$, certain constants $c > 0$, and integer $K_1 \geq K$ appear, in order that $u(k) \geq ck$ for $k \geq K_1$. By summing the Eqn. (2) as of K_1 to ∞ and implementing the last inequality we attain

$$\begin{aligned} & c^{\beta}\sum_{k=0}^{\infty}(K_1+kl)^{\beta}q(K_1+kl) \\ & \leq \sum_{k=0}^{\infty}u(K_1+kl)q(K_1+kl) \\ & \leq a\Delta_{a(l)}(p(K_1)\left|\Delta^2_{a(l)}u(K_1)\right|^{\gamma-1}\Delta^2_{a(l)}u(K_1)) \\ & +(a-1)\sum_{\gamma=0}^{\infty}\Delta_{a(l)}\begin{pmatrix} p(K_1+rl)\left|\Delta^2_{a(l)}u(K_1+rl)\right|^{r-1} \\ \Delta^2_{a(l)}u(K_1+rl) \end{pmatrix} \end{aligned}$$

Therefore, we deduce that

$$\sum_{k=0}^{\infty}(K_1+kl)^{\beta}q(K_1+kl) < \infty \tag{32}$$

Contrarily, in Eqn. (6), there exists an integer $K_2 \geq K_1$, in order that $\psi(K+3l) \leq 1$ for $k \geq K_2$. The fact $\beta \geq 1$, implies $k^{\beta} \geq k$ for $k \geq k_3 = \max\{k_2, 1\}$, and consequently, we attain

$$\begin{aligned} & \sum_{k=0}^{\infty}(K_3+kl)q(K_3+kl)\psi^{\beta}(K_3+(k+3)l) \\ & \leq \sum_{k=0}^{\infty}(K_3+kl)q(K_3+kl) \qquad (33) \\ & \leq \sum_{k=0}^{\infty}(K_3+kl)^{\beta}q(K_3+kl) < \infty \end{aligned}$$

Therefore, Eqn. (32) is a contradiction to Eqn. (31).

Case(II): If a positive solution $u(k)$ is attained from type (II), then from Proposition 10, it is confirmed that the condition (Eqn. (31)) fails to hold.

Case(III): Consider $u(k)$ as a positive solution of Eqn. (2) of type (III) for all $k \geq K$. By multiplying Eqn. (2) by k, and totaling the resultant equation as of K to $k-1$ by implementing the summation by parts, we attain

$$\begin{aligned} & \sum_{S=0}^{k-1}(K+sl)q(K+sl)u^{\beta}(K+(s+3)l) \\ & = c_3 - k\Delta_{a(l)}(p(k)\left|\Delta^2_{a(l)}u(k)\right|^{\gamma-1}\Delta^2_{a(l)}u(k)) \\ & + ap(k+l)\left|\Delta^2_{a(l)}u(k+l)\right|^{\gamma-1}\Delta^2_{a(l)}u(k+l) \\ & -(1-a)\sum_{r=0}^{k}u(K+rl) \\ & \times\Delta_{a(l)}\begin{pmatrix} p(K+rl) \\ \left|\Delta^2_{a(l)}u(K+rl)\right|^{r-1}\Delta^2_{a(l)}u(k+l) \end{pmatrix} \\ & -\sum_{r=0}^{k}(r+a(1+r)l)\begin{pmatrix} p(K+rl) \\ \left|\Delta^2_{a(l)}u(K+rl)\right|^{r-1}\Delta^2_{a(l)}u(K+rl) \end{pmatrix} \\ & -\sum(K+sl)q(K+sl)u^{\beta}(K+(S+3)l) < c_3 \end{aligned}$$

Where

$$\begin{aligned} c_3 &= aK\Delta_{a(l)}\left(p(k)\left|\Delta^2_{a(l)}u(K)\right|^{r-1}\Delta_{a(l)}u(K)\right) \\ & -P(K+l)\left|\Delta^2_{a(l)}u(K+l)\right|^{r-1}\Delta^2_{a(l)}u(K+l) > 0, \end{aligned}$$

is constant. Hence, we infers that

$$\sum_{k=0}^{\infty}(k_0+kl)q(k_0+kl)\psi^{\beta}((k=3)l) < \infty$$

Since $u(k)$ is increasing,

$$\sum_{k=0}^{\infty}(k_0+kl)q(k_0+kl) < \infty \tag{34}$$

By integrating Eqn. (33) and Eqn. (34), the result contradicts the assumption Eqn. (31).

Case (IV): If there is a positive solution $u(k)$ of type (IV), then as of Proposition 12, we infers that Eqn. (31) is not satisfied. This proves the provided theorem.

Theorem 15. *Let* $\beta < 1 \leq \gamma$. *If*

$$\sum_{k=0}^{\infty}\left(\frac{\frac{1}{p(k_0+kl)}}{\sum_{s=0}^{k-1}(k_0+(k-s)l)(k_0+sl)^{\beta}q(k_0+sl)}\right)=\infty \quad (35)$$

then each solution of Eqn. (2) is oscillatory.

Proof: Presume, to contrary, If $u(k)$ is a positive solution of Eqn. (2), then $u(k)$ falls under one of the '4' types (I to IV) previously mentioned in Lemma 1. Therefore, it is mostly enough to confirm that each case contradicts Eqn. (35)."

Case (I): Consider $u(k)$ as a positive solution of Eqn. (2) of type (I) for all $k \geq K$. As in the proof of Theorem 14, we attain Eqn. (32). Then,

$$\begin{aligned}&\sum_{s=0}^{k-1}(k_0+(k-s)l)(k_0+sl)^{\beta}q(k_0+sl)\\ &\leq (k-k_0)\sum_{s=0}^{k-1}(k_0+sl)^{\beta}q(k_0+sl)\\ &\leq Qk \; for \, k \geq K,\end{aligned}$$

Where, $Q=\sum_{k=0}^{\infty}(k_0+kl)^{\beta}q(k_0+kl)$. By implementing the last inequality, we attain

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{1}{p^{\frac{1}{\gamma}}(K+kl)}\left[\sum_{s=0}^{k-1}(k_0+(k-s)l)(k+sl)^{\beta}\right]^{\frac{1}{\gamma}}\\ &\geq Q^{\frac{1}{\gamma}}\sum_{k=0}^{\infty}\left(\frac{K+kl}{p(K+kl)}\right)^{\frac{1}{\gamma}}\end{aligned} \quad (36)$$

In view of Eqn. (6), the Eqn. (36) implies that Eqn. (35) fails to hold.

Case(II): If $u(k)$ is a positive solution of type (II), then as of Proposition 11, we infer that Eqn. (35) is not satisfied."

Case (III): Consider $u(k)$ as a positive solution of Eqn. (2) of type (III) for all $k \geq K$. As in the proof of Theorem 14, we attain Eqn. (34). With the Eqn. (34) and fact $\beta < 1$ which implies $k^{\beta} < k$ for all $k \geq \max\{1, k_0\} = K_1$, we get Eqn. (32). Now, as in case (I), utilizing Eqn. (32), we get a contradiction of Eqn. (35)."

Case (IV): If $u(k)$ is a positive solution of type (IV), then as of Proposition 13, we found that condition (Eqn. (35)) fails to hold. This proves the provided theorem.

IV. APPLICATIONS

In this section, we offered some examples to elucidate the results provided in the former section

Example 16. *Consider the generalized quasi-linear* a *-difference equation*

$$\begin{aligned}&\Delta_{a(l)}^{2}\left(k^{\mu}\left|\Delta_{a(l)}^{2}u(k)\right|^{\gamma-1}\Delta_{a(l)}^{2}u(k)\right)\\ &+k^{-\lambda}\left|u((k+3)l\right|^{\beta-1}u((k+3)l)=0, k\in N(k_0)\end{aligned} \quad (37)$$

Let $\beta \geq 1 > \gamma$ and $\mu > 1+\gamma$. The assumption makes sure that Eqn. (6) is satisfied for the function $p(k)=k^{\mu}$. Then, by utilizing Theorem 14, "the Eqn. (2) is oscillatory if (31) is satisfied", we get

$$\psi(k)=\sum_{s=k}^{\infty}\frac{(k_0+(s-k+l)}{p^{\frac{1}{\gamma}}(s)}\approx k^{2-\frac{\mu}{\gamma}}, k\to\infty,$$

it follows that

$$\begin{aligned}&\sum_{k=0}^{\infty}(k_0+kl)q(k_0+kl)\psi^{\beta}((k+3l)l)\\ &\approx\sum_{k=0}^{\infty}(k_0+kl)^{1-\lambda+\beta\left(2-\frac{\mu}{\gamma}\right)}, k\to\infty\end{aligned}$$

Therefore, the condition Eqn. (31) holds if $2-\lambda+\beta(2-\frac{\mu}{\gamma})>0$. Hence, Eqn. (37) is oscillatory, if $\lambda < 2+\beta(2-\frac{\mu}{\gamma})>0$. Utilizing the presumptions $\beta \geq 1 > \gamma$ and $\mu > 1+\gamma$, we have $\lambda < 2+(1-\frac{1}{\gamma})\beta < 2$

Consider $\beta < 1 \leq \gamma$. The assumption $\mu > 2\gamma$ makes sure that Eqn. (6) is satisfied.Then, by utilizing Theorem 15, we attain that the Eqn. (37) is oscillatory, if condition Eqn. (35) is satisfied. For $q(k)=k^{-A}$, it is simple that

$$\sum_{S=0}^{K-1}\sum_{t=0}^{s-1}(k_0+tl)^{\beta}q(k_0+tl)\approx k^{\beta-\lambda+2},k\to\infty,$$

and so,

$$\sum_{k=0}^{\infty}\left(\frac{1}{p(k_0+kl)}\sum_{s=0}^{k-1}(k_0+(k-s)l)(k_0+sl)q(k_0+sl)\right)^{\frac{1}{\gamma}}$$

$$\approx\sum_{k=0}^{\infty}(k_0+kl)^{\frac{\beta-\lambda+2-\mu}{\gamma}},k\to\infty$$

Then the condition Eqn. (35) holds if $\lambda<\gamma+\beta-\mu+2$. Therefore, the Eqn. (37) is oscillatory if $\lambda<\gamma+\beta-\mu+2$. Using the presumptions $\beta<1\le\gamma$, $\mu>2\gamma$ and, we attain that $\lambda<\beta-\gamma+2<2$.

Example 17. *Consider the generalized quasilinear a -difference equation*

$$\Delta^2_{a(l)}\left(2^k(\Delta^2_{a(l)}u(k))^{\frac{1}{3}}\right)+2^{3k+9}9^{\frac{2}{3}}u((k+3)l)=0,k\ge1 \quad (38)$$

Here $p(k)=2^k,q(k)=2^{3k+9}9^{\frac{2}{3}},\gamma=\frac{1}{3},\beta=1$.It is perceived that all conditions of Theorem 14 are satisfied and thereby each solution of Eqn. (38) is oscillatory.

In fact, $u(k)=\left\{\frac{(-1)^k}{2^3k}\right\}$ is one such solution of Eqn. (38).

Example 18. *Consider the generalized quasilinear a - difference equation*

$$\Delta^2_{a(l)}(k^7 1(\Delta^2_{a(l)}u(k))^3)$$
$$+64\left((k+2)^7+2(k+1)^7+k^7\right)u^{\frac{1}{3}}((k+3)l)=0,k\ge1 \quad (39)$$

Here $p(k)=k^7,q(k)=64((k+2)^7+2(k+1)^7+k^7)$, $\gamma=3,\beta=\frac{1}{3}$. It is perceived that all conditions of Theorem 15 are satisfied and thereby each solutions of Eqn. (39) is oscillatory. In fact $u(k)=\left\{(-1)^k\right\}$ is one such solution of Eqn. (39).

This paper has a subsequent remark.

Remark 19. The results attained in this paper offers a partial answer to the problem provided in [22]. Further, the results reduce to that of in [24] when $\gamma=1$.

V. RESULT ANALYSIS

This paper proved the classification of non-oscillatory positive solutions of ***FOQGDEs*** which holds four types of cases. But to the best of the knowledge, no research has discussed the positive solution of generalized a- difference equations. Utilizing these results, we found that the generalized quasi-linear a - difference equations are either oscillatory or non-oscillatory in a simple way by utilizing its positive solutions with certain conditions.

VI. CONCLUSION

We derived the sufficient conditions for oscillations of each solution of the ***FOQGDEs*** in this work. Also, results are validated and proved with appropriate examples by using MATLAB.

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