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Certain Identities on Class one Infinite Series

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Article Info
Volume 82
Page Number: 7512 - 7523
Publication Issue:
January-February 2020

Article History
Article Received: *18 May 2019*
Revised: *14 July 2019*
Accepted: *22 December 2019*
Publication*: 03 February 2020*

Abstract

Ramanujan recorded different classes of beautiful infinite series in his lost notebook and presented a relation of these series with Eisenstein series. Shaun cooper established identities involving Eisenstein series and modular forms and functions of weight one and weight two. In this paper, we establish certain identities involving the infinite series with modular forms and functions of weight one and weight two. Also, the convolution sum have been evaluated using Eisenstein series of level 3 and 6 recorded by Shaun Cooper.

Keywords: *Eisenstein series, Dedekind -function, Convolution sum, 2010 Mathematics Subject Classification: 11M36, 11F20*

1 INTRODUCTION

Ramanujan, in his paper [14],[17, p. 136-162] recorded several noticeable theorems involving Eisenstein series and claims that different classes of infinite series may be exhibited in terms of Eisenstein series. Also, in his Lost Notebook [16], Ramanujan documented certain identities relating the class one infinite series $T_{2k}(q)$, $q=1, 2,...,6$ and second class infinite series $U_n(q)$, $n= 0, 2, 4, 6, 8, 10$ with the Eisenstein series $P(q)$, $Q(q)$ and $R(q)$. The first proof of these six identities for $T_{2k}(q)$ published in a paper by B. C. Berndt and A. J. Yee [8]. Another version of these relations presented in a paper by Z. -G. Liu's[13, p. 9-12]. The remaining six formulas for second class infinite series $U_n(q)$ stated by Ramanujan, were established by B. C. Berndt [9], [10], employing well known Jacobi's identity, $q-$series [7] and the differential equations recorded by Ramanujan [16]. Further, S. Cooper [6], proved certain identities involving Eisenstein series of levels 2, 3, 4, 6 and the weight one and two modular forms and functions.

The present study establishes certain identities that involve the infinite series $T_2(q^n)$ and $U_2(q^n)$ for $n=2,3,4$ and 6 and relations among $T_2(q^n)$ for $n=1, 2, 3, 4, 6$ and the modular forms of weight one and two, deduced by Cooper.Also, this study provides an adequate method to evaluate convolution sums, which is achieved by adopting some of the Eisenstein series relations recorded by Cooper and Glaisher.Section 2 is dedicated to record some preliminary results.

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2 PRELIMINARIES

In the upper half plane $L=\{\tau:\ Im(\tau)>0\ \}$, with $q=e^{2\pi i\tau}$, the Dedekind η-function is generally represented by

$$\eta_k(\tau):=q^{k/24}\prod_{r=1}^{\infty}\left(1-q^{kr}\right),\quad |q|<1.$$

The class one infinite series

$$T_{2l}(q):=1+\sum_{r=1}^{\infty}(-1)^r\left[(6r-1)^{2l}q^{\frac{r(3r-1)}{2}}+(6r+1)^{2l}q^{\frac{r(3r+1)}{2}}\right],\quad |q|<1\ (1)$$

introduced by Ramanujan, in his Lost Notebook [16] and deduced a relation among this infinite series and Ramanujan-type Eisenstein series defined by

$$P(q):=1-24\sum_{j=1}^{\infty}\frac{jq^j}{1-q^j}=1+24q\frac{d}{dq}\sum_{n=1}^{\infty}log(1-q^j)\qquad(2)$$

$$Q(q):=1+240\sum_{j=1}^{\infty}\frac{j^3q^j}{1-q^j}\ \text{and}\ R(q):=1-504\sum_{j=1}^{\infty}\frac{j^5q^j}{1-q^j}$$

for $l=1,2,...,6.$ Furthermore, B. C. Berndt [10] establish the relation

$$T_2(q)=(q;q)_\infty P(q),\qquad(3)$$

where

$$(q;q)_\infty:=f_1:=f(-q)=q^{-1/24}\eta(\tau)=\prod_{j=1}^{\infty}(1-q^j)=1+\sum_{j=1}^{\infty}(-1)^j\{q^{\frac{j(3j-1)}{2}}+q^{\frac{j(3j+1)}{2}}\}.\qquad \text{is}$$

recognized as a famous pentagonal number theorem [7]. Further, Ramanujandocumented the second class of infinite series in Chapter 16 of his Second Notebook [15], namely

$$U_n(q):=\frac{1}{(q;q)_\infty^3}\sum_{j=1}^{\infty}(-1)^{j-1}(2j-1)^{n+1}q^{\frac{j(j-1)}{2}},\ n\in\mathbb{N}^+$$

where

$$(q;q)_\infty^3=\frac{1}{2}\sum_{j=-\infty}^{\infty}(-1)^j(2j+1)q^{\frac{j(j+1)}{2}},$$

called the Jacobi's identity [7, p. 39].FurtherRamanujan [16, p. 369] recorded the differential recurrence relation for second class of infinite series, namely

$$U_{r+2}(q)=P(q)U_r(q)+8qU_r^{'}(q),\qquad r\in\mathbb{N}^+.\ (4)$$

Later B. C. Berndt [10] proved the identity $U_0(q)=1$ and $U_2(q)=P(q)$. Next, we define modular forms and functions of weight one and weight two, as recorded by S. Cooper [6]. The weight one modular forms z_a, z_b, z_c and z_d are studied in conjunction with the modular functions x_a, x_b and x_c expressed in terms of Dedekind η-functionidentities are listed below:

$$z_a=\frac{\eta_1^6\eta_6}{\eta_2^3\eta_3^2},\ z_b=\frac{\eta_2^6\eta_3}{\eta_1^3\eta_6^2},\ z_c=\frac{\eta_2\eta_3^6}{\eta_1^2\eta_6^3},\ z_d=\frac{\eta_2\eta_3^6}{\eta_1^2\eta_6^3}.$$

$$x_a=\frac{\eta_2\eta_6^5}{\eta_1^5\eta_3},\ x_b=\frac{\eta_1^4\eta_6^8}{\eta_2^8\eta_3^4},\ x_c=\frac{\eta_1^3\eta_6^9}{\eta_2^3\eta_3^9}.$$

The weight two modular forms y_a, y_b, y_c and the modular functions w_a, w_b, w_c are listed below:

$$y_a=q\frac{d}{dq}logx_a,\ y_b=q\frac{d}{dq}logx_b,\ y_c=q\frac{d}{dq}logx_c.$$

$$w_a=\frac{\eta_1^{12}\eta_6^{12}}{\eta_2^{12}\eta_3^{12}},\ w_b=\frac{\eta_2^6\eta_6^6}{\eta_1^6\eta_3^6},\ w_c=\frac{\eta_3^4\eta_6^4}{\eta_1^4\eta_2^4}.$$

Also, in his book [6], Cooper recorded and proved certain identities involving these modular forms, modular functions and the Ramanujan-type Eisenstein series $P(q^n)$ for $n=1,2,3$ and 6.

Lemma 2.1 *[6]* The following series expansions hold:

$$z_a-2z_b=P(q)-2P(q^2), \tag{5}$$

$$(z_a+3z_d)^2=-P(q)+3P(q^3), \tag{6}$$

$$2(1+6x_a)^2z_a^2=-P(q^2)+3P(q^6), \tag{7}$$

$$\left[5-\frac{x_a}{(1+8x_a)(1+9x_a)}\right]^2 z_b^2z_c^2=-P(q)+6P(q^6). \tag{8}$$

Lemma 2.2 *[6]* The following relation holds:

$$z_a^2=\frac{1}{2}\left(P(q)-8P(q^2)+9P(q^3)\right),$$

$$z_b^2=\frac{1}{8}\left(-2P(q)+P(q^2)+9P(q^6)\right),$$

$$z_c^2=\frac{1}{6}\left(-P(q)-P(q^3)+8P(q^3)\right),$$

$$z_d^2=\frac{1}{24}\left(-P(q^2)+2P(q^3)-P(q^6)\right).$$

Lemma 2.3 *[6]* The following relation holds:

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$$(-1+72x_a^2)z_a^2=\frac{1}{2}\left(-P(q)+2P(q^2)+3P(q^3)-6P(q^6)\right),$$

$$(-1+9x_b^2)z_b^2=\frac{1}{4}\left(P(q)-2P(q^2)+3P(q^3)-6P(q^6)\right),$$

$$(-1-8x_c^2)z_c^2=\frac{1}{6}\left(P(q)+2P(q^2)-3P(q^3)-6P(q^6)\right).$$

Lemma 2.3 *[6]* The following relation holds: \

$$z_az_b=-\frac{1}{8}\left(-P(q)+2P(q^2)+9P(q^3)-18P(q^6)\right),$$

$$z_az_c=-\frac{1}{6}\left(-P(q)+4P(q^2)+3P(q^3)-12P(q^6)\right),$$

$$z_bz_c=-\frac{1}{24}\left(5P(q)-2P(q^2)+3P(q^3)-30P(q^6)\right),$$

$$z_az_d=-\frac{1}{24}\left(P(q)-10P(q^2)+15P(q^3)-6P(q^6)\right),$$

$$z_bz_d=-\frac{1}{24}\left(P(q)-P(q^2)-3P(q^3)+3P(q^6)\right),\ z_cz_d=-\frac{1}{24}\left(P(q)-2P(q^2)-P(q^3)+2P(q^6)\right).$$

Lemma 2.5 *[16]*The following identities hold:

$$(1-5w_a)y_a=3P(q^3)-2P(q^2),$$

$$(5-w_a)y_a=6P(q^6)-P(q),$$

$$(1+16w_b)y_b=\frac{1}{2}(3P(q^3)-P(q)),$$

$$(1+4w_b)y_b=\frac{1}{2}(3P(q^6)-P(q^2)),$$

$$(1+27w_c)y_c=2P(q^2)-P(q),$$

$$(1+3w_c)y_c=2P(q^6)-P(q^3).$$

Lemma 2.6 *[16]*The following identities hold:

$$A_aw_a\frac{dy_a}{dw_a}=-\frac{1}{24}\left(5P(q)-14P(q^2)-21P(q^3)+30P(q^6)\right),$$

$$A_bw_b\frac{dy_b}{dw_b}=-\frac{1}{6}\left(-P(q)+P(q^2)-3P(q^3)+3P(q^6)\right),$$

$$A_cw_c\frac{dy_c}{dw_c}=-\frac{1}{8}\left(-P(q)-2P(q^2)+P(q^3)+2P(q^6)\right),$$

where

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$$A_a = \sqrt{1-34w_a + w_a^2},\ A_b = \sqrt{1+20w_b + 64w_b^2} \text{ and } A_c = \sqrt{1+14w_c + 81w_c^2}.$$

3 MAIN RESULTS

Theorem 3.1 For any positive integer $n \geq 2$, we have

$$P(q^n) = 1 + nq\left[\frac{T_2(q^n)+1}{(q^n;q^n)_\infty} - 1\right].$$

Proof First we prove the result for $n = 2$. Replacing q to q^2 in (2), we obtain

$$P(q^2) = 1 + 24q\frac{d}{dq}\sum_{n=1}^{\infty} log(1-q^{2n})$$

$$= 1 + 24q\frac{d}{dq}log(q^2;q^2)$$

$$= 1 + 24q^2\frac{1}{(q^2;q^2)_\infty}\frac{d}{dq}(q^2;q^2)_\infty$$

On simplifying, we get

$$(q^2;q^2)_\infty P(q^2) = (q^2;q^2)_\infty + 24q^2\frac{d}{dq}\left[1+\sum_{r=1}^{\infty}(-1)^r\{q^{r(3r-1)} + q^{r(3r+1)}\}\right]$$

$$= (q^2;q^2)_\infty + 24q\sum_{r=1}^{\infty}(-1)^r\left[r(3r-1)q^{r(3r-1)} + r(3r+1)q^{r(3r+1)}\right]$$

$$= (q^2;q^2)_\infty + 2q\sum_{r=1}^{\infty}(-1)^r\left[((6r-1)^2-1)q^{r(3r-1)} + ((6r+1)^2+1)q^{r(3r+1)}\right]$$

$$-2q(q^2;q^2)_\infty + 2q$$

$$= (q^2;q^2)_\infty + 2qT_2(q^2) - 2q(q^2;q^2)_\infty + 2q.$$

Dividing throughout by $(q^2;q^2)_\infty$ and then, by rearranging the terms, we deduce the result for $n = 2$. Similarly, the proof of $n > 2$ follows by replacing q to q^n in (2) and using the series (1).

Theorem 3.2 For every integer $m \geq 2$, we deduce

$$U_0(q^m) = 1, \tag{9}$$

$$U_2(q^m) = P(q^m). \tag{10}$$

*Proof*The identity (9) holds by putting $n = 0$ and changing q to q^m for $m \geq 2$ in (4). The equalities (10) follows directly, by replacing q to q^m for $m \geq 2$, and $n = 2$ in (4) and using (9).

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Theorem3.3 The following identities among the class one and second class infinite series holds:

$$qT_2(q^2)-f_2U_2(q^2)+2q\left(1-f_2+\frac{1}{2q}\right)=0,$$

$$3q^2T_2(q^3)-f_3U_2(q^3)+3q^2\left(1-f_3+\frac{1}{3q^2}\right)=0,$$

$$4q^3T_2(q^4)-f_4U_2(q^4)+4q^3\left(1-f_4+\frac{1}{4q^3}\right)=0,$$

$$6q^5T_2(q^6)-f_6U_2(q^6)+6q^5\left(1-f_6+\frac{1}{6q^5}\right)=0.$$

*Proof*Prooffollows directly, by eliminating $P(q^n)$ between Theorem 3.1 (i)-(iv) and (10) for $n=2,3,4$ and 6.

Theorem 3.4 The following equations hold:

$$\frac{T_2(q)}{f_1}-4q\frac{T_2(q^2)}{f_2}+4q\left(1-\frac{1}{f_2}\right)+(1+27w_c)y_c-2=0,$$

$$\frac{T_2(q)}{f_1}-9q^2\frac{T_2(q^3)}{f_3}+9q^2\left(1-\frac{1}{f_3}\right)+2(1+16w_b)y_b-3=0,$$

$$4\frac{T_2(q^2)}{f_2}-9q\frac{T_2(q^3)}{f_3}+9q\left(1-\frac{1}{f_3}\right)-2\left(1-\frac{1}{f_2}\right)+\frac{(1-5w_a)y_a}{q}-\frac{1}{q}=0,$$

$$\frac{T_2(q)}{f_1}-36q^5\frac{T_2(q^6)}{f_6}+36q^5\left(1+\frac{1}{f_6}\right)+(5-w_a)y_a-6=0,$$

$$\frac{T_2(q^2)}{f_2}-9q^4\frac{T_2(q^6)}{f_6}+9q^4\left(1+\frac{1}{f_6}\right)-\left(1-\frac{1}{f_2}\right)+\frac{(1+4w_b)y_b}{q}-\frac{1}{q}=0,$$

$$\frac{T_2(q^3)}{f_3}-4q^3\frac{T_2(q^6)}{f_6}+4q^3\left(1-\frac{1}{f_6}\right)-\left(1-\frac{1}{f_3}\right)+\frac{(1+3w_c)y_c}{3q^2}-\frac{1}{3q^2}=0.$$

Proof These identities follows directly by employing (3), Theorem 3.1 (i), (ii) and (iv) in Lemma 2.5.

Theorem 3.5 The following series expansions hold:

$$\frac{T_2(q)}{f_1}-16q\frac{T_2(q^2)}{f_2}+27q^2\frac{T_2(q^3)}{f_3}-27q^2\left(1-\frac{1}{f_3}\right)-16q\left(1+\frac{1}{f_2}\right)+1-2z_a^2=0,$$

$$\frac{T_2(q)}{f_1}-q\frac{T_2(q^2)}{f_2}-27q^5\frac{T_2(q^6)}{f_6}+54q^5\left(1-\frac{1}{f_6}\right)+q\left(1-\frac{1}{f_2}\right)-8z_b^2=0,$$

$$\frac{T_2(q)}{f_1}-21q^2\frac{T_2(q^3)}{f_3}-21q^2\left(1-\frac{1}{f_3}\right)-6z_c^2+7=0,$$

$$\frac{T_2(q^2)}{f_2}-3q\frac{T_2(q^3)}{f_3}+3q^4\frac{T_2(q^6)}{f_6}-6q^3\left(1-\frac{1}{f_6}\right)+3q\left(1-\frac{1}{f_3}\right)-\left(1-\frac{1}{f_2}\right)+\frac{12}{q}z_d^2=0.$$

Proof Using (3), Theorem 3.1 (i), (ii) and (iv) in Lemma 2.2 and then rearranging the terms and simplifying, we arrive at the required result.

Theorem 3.6The following relations hold:

$$5\frac{T_2(q)}{f_1}-28q\frac{T_2(q^2)}{f_2}-63q^2\frac{T_2(q^3)}{f_3}+180q^5\frac{T_2(q^6)}{f_6}-180q^5\left(1-\frac{1}{f_6}\right)$$

$$+63q^2\left(1-\frac{1}{f_3}\right)+28q\left(1-\frac{1}{f_2}\right)-24A_a w_a\frac{dy_a}{dw_a}-5=0,$$

$$\frac{T_2(q)}{f_1}-2q\frac{T_2(q^2)}{f_2}+9q^2\frac{T_2(q^3)}{f_3}-18q^5\frac{T_2(q^6)}{f_6}+18q^5\left(1-\frac{1}{f_6}\right)$$

$$-9q^2\left(1-\frac{1}{f_3}\right)+2q\left(1-\frac{1}{f_2}\right)-6A_b w_b\frac{dy_b}{dw_b}-1=0,$$

$$\frac{T_2(q)}{f_1}+4q\frac{T_2(q^2)}{f_2}-3q^2\frac{T_2(q^3)}{f_3}-12q^5\frac{T_2(q^6)}{f_6}+12q^5\left(1-\frac{1}{f_6}\right)$$

$$+3q^2\left(1-\frac{1}{f_3}\right)-4q\left(1-\frac{1}{f_2}\right)-8A_c w_c\frac{dy_c}{dw_c}-1=0.$$

Proof On using (3), Theorem 3.1 (i), (ii) and (iv) in Lemma 2.6 and then simplifying, we deduce the required result.

Theorem 3.7 The following equality holds:

$$\frac{T_2(q)}{f_1}-4q\frac{T_2(q^2)}{f_2}-9q^2\frac{T_2(q^3)}{f_3}+36q^5\frac{T_2(q^6)}{f_6}+36q^5\left(1-\frac{1}{f_6}\right)$$

$$+9q^2\left(1-\frac{1}{f_3}\right)+4q\left(1-\frac{1}{f_2}\right)-2(1-72x_a^2)z_a^2+1=0,$$

$$\frac{T_2(q)}{f_1}-4q\frac{T_2(q^2)}{f_2}+9q^2\frac{T_2(q^3)}{f_3}-36q^5\frac{T_2(q^6)}{f_6}+36q^5\left(1-\frac{1}{f_6}\right)$$

$$-9q^2\left(1-\frac{1}{f_3}\right)+4q\left(1-\frac{1}{f_2}\right)+4(1-9x_b^2)z_b^2-5=0,$$

$$\frac{T_2(q)}{f_1}+4q\frac{T_2(q^2)}{f_2}-9q^2\frac{T_2(q^3)}{f_3}-36q^5\frac{T_2(q^6)}{f_6}+36q^5\left(1-\frac{1}{f_6}\right)$$

$$+9q^2\left(1-\frac{1}{f_3}\right)-4q\left(1-\frac{1}{f_2}\right)+6(1+8x_c^2)z_c^2-4=0,$$

$$\frac{T_2(q)}{f_1}-4q\frac{T_2(q^2)}{f_2}-27q^2\frac{T_2(q^3)}{f_3}+108q^5\frac{T_2(q^6)}{f_6}-108q^5\left(1-\frac{1}{f_6}\right)$$

$$+27q^2\left(1-\frac{1}{f_3}\right)+4q\left(1-\frac{1}{f_2}\right)+8z_az_b+7=0,$$

$$\frac{T_2(q)}{f_1}-8q\frac{T_2(q^2)}{f_2}-9q^2\frac{T_2(q^3)}{f_3}+72q^5\frac{T_2(q^6)}{f_6}-72q^5\left(1-\frac{1}{f_6}\right)$$

$$+9q^2\left(1-\frac{1}{f_3}\right)+8q\left(1-\frac{1}{f_2}\right)-6z_az_c+5=0,$$

$$5\frac{T_2(q)}{f_1}-4q\frac{T_2(q^2)}{f_2}+9q^2\frac{T_2(q^3)}{f_3}-180q^5\frac{T_2(q^6)}{f_6}+180q^5\left(1-\frac{1}{f_6}\right)$$

$$-3q^2\left(1-\frac{1}{f_3}\right)+4q\left(1-\frac{1}{f_2}\right)-24z_bz_c-29=0,$$

$$\frac{T_2(q)}{f_1}-20q\frac{T_2(q^2)}{f_2}+45q^2\frac{T_2(q^3)}{f_3}-36q^5\frac{T_2(q^6)}{f_6}+36q^5\left(1-\frac{1}{f_6}\right)$$

$$-45q^2\left(1-\frac{1}{f_3}\right)+20q\left(1-\frac{1}{f_2}\right)+24z_az_d-1=0,$$

$$\frac{T_2(q)}{f_1}-2q\frac{T_2(q^2)}{f_2}-9q^2\frac{T_2(q^3)}{f_3}+18q^5\frac{T_2(q^6)}{f_6}-18q^5\left(1-\frac{1}{f_6}\right)$$

$$+9q^2\left(1-\frac{1}{f_3}\right)+2q\left(1-\frac{1}{f_2}\right)+24z_bz_d-1=0,$$

$$\frac{T_2(q)}{f_1}-4q\frac{T_2(q^2)}{f_2}-3q^2\frac{T_2(q^3)}{f_3}+12q^5\frac{T_2(q^6)}{f_6}-12q^5\left(1-\frac{1}{f_6}\right)$$

$$+3q^2\left(1-\frac{1}{f_3}\right)+4q\left(1-\frac{1}{f_2}\right)+24z_cz_d-1=0.$$

Proof Using Theorem 3.1 (i), (ii), (iv) and (3), in Lemma 2.3 and 2.4 and then rearranging the terms and simplifying, we arrive at the required result.

4 APPLICATION TO CONVOLUTION SUM

Definition4.1For $a,b \in \mathbb{N}$, the convolution sum is defined by

$$U_{a,b}(m) := \sum_{ai+bj=m} \sigma(i)\sigma(j),$$

where $a \le b$ and for any $l,m \in \mathbb{N}$, $\sigma_l(m) = \sum_{u/m} u^l$ and $\sigma_l(m) = 0$ for $m \notin \mathbb{N}$. For every nonnegative m, the convolution sum $\sum_{r+ks=m} \sigma(r)\sigma(s)$ has been assessed explicitly for $s = 1-9,12,16,18$ and 24, by A. Alaca et. al.[1-5] and K. S. Williams et. al. [18,19]. Also E. X. W. Xia and O. X. M. Yao [20] have determined the illustrations for $\sum_{r+6s=m} \sigma(r)\sigma(s)$ and $\sum_{r+12s=m} \sigma(r)\sigma(s)$. Our proofs are simple and elementary andkeys to our proofs are the claims of J. W. L. Glaisher[11,12],

$$P^2(q) = 1 + \sum_{l=1}^{\infty} (240\sigma_3(l) - 288l\sigma(l))q^l. \qquad (11)$$

Theorem 4.2 For any $n \in \mathbb{N} - \{0\}$, the following identities hold:

$$i) \sum_{2r+3s=n} \sigma(r)\sigma(s) = -\frac{1}{24}\sigma\left(\frac{n}{2}\right) - \frac{5}{36}n\sigma_3\left(\frac{n}{2}\right) + \frac{1}{12}n\sigma\left(\frac{n}{2}\right) - \frac{13}{192}n\sigma\left(\frac{n}{3}\right)$$

$$-\frac{5}{16}n\sigma_3\left(\frac{n}{3}\right) + \frac{1}{8}n\sigma\left(\frac{n}{3}\right) - \frac{1}{20736}[A(n) - B(n) + 3C(n)],$$

$$ii) \sum_{3r+6s=n} \sigma(r)\sigma(s) = -\frac{5}{54}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{18}n\sigma\left(\frac{n}{2}\right) - \frac{5}{48}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{24}n\sigma\left(\frac{n}{3}\right)$$

$$-\frac{1}{12}n\sigma\left(\frac{n}{6}\right) - \frac{1}{36}\sigma\left(\frac{n}{2}\right) + \frac{1}{192}\sigma\left(\frac{n}{3}\right) + \frac{1}{24}\sigma\left(\frac{n}{6}\right) + \frac{1}{12}A(n)$$

$$-\frac{1}{3}B(n) + \frac{13}{12}C(n) - \frac{1}{5184}D(n) + 4E(n) - \frac{1}{4}F(n),$$

where

$$\sum_{n=1}^{\infty} A(n)q^n = z_a^4, \quad \sum_{n=1}^{\infty} B(n)q^n = (z_a - 2z_b)^2, \sum_{n=1}^{\infty} C(n)q^n = (z_b + 3z_d)^4, \quad \sum_{n=1}^{\infty} D(n)q^n = (1-72x_a^2)^2 z_a^4,$$

$$\sum_{n=1}^{\infty} E(n)q^n = (1+6x_a)^4 z_a^4, \qquad \sum_{n=1}^{\infty} F(n)q^n = (5-w_a)^2 z_b^2 z_c^2 = \left[5 - \frac{x_a}{(1+8x_a)(1+9x_a)}\right]^2 z_b^2 z_c^2.$$

Proof i) On squaring the first identity of Lemma 2.2, we get

$$P^2(q) + 6P^2(q^2) + 81P^2(q^3) - 16P(q)P(q^2) + 18P(q)P(q^3) - 144P(q^2)P(q^3) = 4z_a^2.$$

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Now employing (11) and the definition of $P(q^l)$ and then comparing the coefficients of q^n, we obtain

$$48\sigma(n)-72n\sigma(n)+960\sigma\left(\frac{n}{2}\right)+3840\sigma_3\left(\frac{n}{2}\right)-2304n\sigma\left(\frac{n}{2}\right)+4860\sigma_3\left(\frac{n}{3}\right)-1944n\sigma\left(\frac{n}{3}\right)$$

$$756\sigma\left(\frac{n}{3}\right)-2304\sum_{r+2s=n}\sigma(r)\sigma(s)+2592\sum_{r+3s=n}\sigma(r)\sigma(s)+20736\sum_{2r+3s=n}\sigma(r)\sigma(s)=A(n), \quad (12)$$

where

$$\sum_{n=1}^{\infty}A(n)q^n=z_a^4.$$

On squaring (5), using the identity (11) and employing the definition of $P(q^l)$ and then comparing the coefficients of q^n on either sides, we deduce

$$\sum_{r+2s=n}\sigma(r)\sigma(s)=\frac{5}{48}\sigma_3(n)-\frac{1}{8}n\sigma(n)+\frac{5}{12}n\sigma_3\left(\frac{n}{2}\right)-\frac{1}{4}n\sigma\left(\frac{n}{2}\right)$$
$$+\frac{1}{24}\sigma\left(\frac{n}{2}\right)+\frac{1}{24}\sigma(n)-\frac{1}{2304}B(n). \quad (13)$$

where

$$\sum_{n=1}^{\infty}B(n)q^n=(z_a-2z_b)^2.$$

On squaring (6), utilizing the identity (11) and employing the definition of $P(q^l)$ and then comparing the coefficients q^n on either sides, we derive

$$\sum_{r+3s=n}\sigma(r)\sigma(s)=\frac{1}{24}\sigma(n)+\frac{5}{72}\sigma_3(n)-\frac{1}{12}n\sigma(n)+\frac{1}{24}\sigma\left(\frac{n}{3}\right)-\frac{1}{4}n\sigma\left(\frac{n}{3}\right)$$
$$+\frac{5}{8}n\sigma_3\left(\frac{n}{3}\right)-\frac{1}{864}C(n). \quad (14)$$

where

$$\sum_{n=1}^{\infty}C(n)q^n=(z_b+3z_d)^4.$$

Now substituting (13) and (14) in (12), and on simplifying, we obtain the required result.

ii) On squaring the first identity of Lemma 2.3, we obtain

$$P^2(q)+4P^2(q^2)+9P^2(q^3)+36P^2(q^6)-4P(q)P(q^2)-6P(qP(q^3)+12P(q)P(q^6)$$
$$+12P(q^2)P(q^3)-24P(q^2)P(q^6)=(1-72x_a^2)^2z_a^4.$$

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Now utilizing the identity (11), employing the definition of $P(q^l)$ and then comparing the coefficients of q^n, we deduce

$$\begin{aligned}
&60\sigma_3(n)-72n\sigma(n)+240\sigma_3\left(\frac{n}{2}\right)-144n\sigma\left(\frac{n}{2}\right)+540\sigma_3\left(\frac{n}{3}\right)\\
&-216n\sigma\left(\frac{n}{3}\right)+2160\sigma_3\left(\frac{n}{6}\right)-432n\sigma\left(\frac{n}{6}\right)-12\sigma(n)+96\sigma\left(\frac{n}{2}\right)\\
&+180\sigma\left(\frac{n}{3}\right)+288\sigma\left(\frac{n}{6}\right)-576\sum_{r+2s=n}\sigma(r)\sigma(s)-5184\sum_{3r+6s=n}\sigma(r)\sigma(s)\\
&-864\sum_{r+3s=n}\sigma(r)\sigma(s)+1728\sum_{r+6s=n}\sigma(r)\sigma(s)+1728\sum_{2r+3s=n}\sigma(r)\sigma(s)\quad-3456\sum_{2r+6s=n}\sigma(r)\sigma(s)=D(n).
\end{aligned} \tag{15}$$

where

$$\sum_{n=1}^{\infty}D(n)q^n=(1-72x_a^2)^2z_a^4.$$

Squaring (7), using the identity (11) and employing the definition of $P(q^l)$ and then comparing the coefficients of q^n, we deduce

$$\begin{aligned}
&\sum_{2r+6s=n}\sigma(r)\sigma(s)=\frac{5}{72}\sigma_3\left(\frac{n}{2}\right)-\frac{1}{24}n\sigma\left(\frac{n}{2}\right)+\frac{5}{8}\sigma_3\left(\frac{n}{6}\right)-\frac{1}{8}n\sigma\left(\frac{n}{6}\right)\\
&+\frac{1}{24}\sigma\left(\frac{n}{2}\right)+\frac{1}{24}\sigma\left(\frac{n}{2}\right)+\frac{1}{6}\sigma(n)-\frac{1}{864}E(n).
\end{aligned} \tag{16}$$

where

$$\sum_{n=1}^{\infty}E(n)q^n=(1+6x_a)^4z_a^4.$$

Squaring (8) using the identity (11) and employing the definition of $P(q^l)$ and then comparing the coefficients of q^n, we arrive at

$$\begin{aligned}
&\sum_{r+6s=n}\sigma(r)\sigma(s)=\frac{5}{144}\sigma_3(n)-\frac{1}{24}n\sigma(n)+\frac{5}{4}\sigma_3\left(\frac{n}{6}\right)-\frac{1}{4}n\sigma\left(\frac{n}{6}\right)\\
&\qquad+\frac{1}{24}\sigma(n)+\frac{1}{24}\sigma\left(\frac{n}{6}\right)-\frac{1}{6912}F(n).
\end{aligned} \tag{17}$$

where

$$\sum_{n=1}F(n)q^n=\left[5-\frac{x_a}{(1+8x_a)(1+9x_a)}\right]^2z_b^2z_c^2.$$

Substituting Theorem 4.2 (i), (13), (14), (16) and (17) in (15) and simplifying, we arrive at the required result.

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