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A Special Representation of Fibonacci polynomials

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Abstract:
In this we shall we a special representation of this Fibonacci polynomials sequence which is defined by a recurrence relation with initial terms. A recurrence relation is very useful to solve many problems in mathematics. Fibonacci polynomials are very useful in mathematics as well as physics. Thinking of famous mathematician Carl Friedrich Gauss (1777– 1855) about number theory: "Mathematics is the queen of all sciences, and Number Theory is the queen of Mathematics."Fibonacci polynomial is very important topic of mathematic many real life problems can be solved by Fibonacci polynomials and Fibonacci numbers.
In Number Theory [1,2] we work on numbers in mathematics many types of numbers for examples Even number, Odd number, prime number, complete square number etc. In Number Theory we want a solution in integers [2,3,7]. Many basic theorems have proved in Number Theory. There many representations of Fibonacci polynomials in number theory we are also giving a special representation of Fibonacci polynomials.

Keywords:-*Sequence, recurrence relation, number theory, coefficient, property.*

Introduction

In number theory, Fibonacci polynomials are defines a sequence; each term of the sequence is depending on the preceding terms. Fibonacci numbers are very useful in number theory. Some real life problems can be solved by Fibonacci numbers. Discovery of Fibonacci numbers by real life problem example. Same as Fibonacci numbers also Fibonacci polynomials are also very useful in number theory. Fibonacci polynomials are sequence of polynomials which find by a special type formula with given initials terms. In number theory or mathematics many representation of the Fibonacci numbers and polynomials in this paper we are also giving a special type representation of Fibonacci polynomials for so some special type's values of variable. By this representation of Fibonacci polynomials we can find the value of any Fibonacci polynomial at any value of variable which satisfied the condition of given representation. Fibonacci numbers and Fibonacci polynomials are depend on recurrence relation and [4, 5 and 6] we classify recurrence relations the number of previous terms needed to find the new term.

First Order Recurrence Relation

In the first order recurrence relation only one initial term is given. For example

$$a_{n+1} = a_n + 5, n \geq 1, a_0 = 0$$

we can find the terms

$$a_1 = 6, a_2 = 7, a_3 = 8$$

Second Order Recurrence Relation

In the second order recurrence relation new term depend on two previous terms and two initial terms are given.
For example

$$a_n = a_{n-1} + 2a_{n-2}, n \geq 2$$

with the initial terms $a_0 = 0, a_1 = 1$

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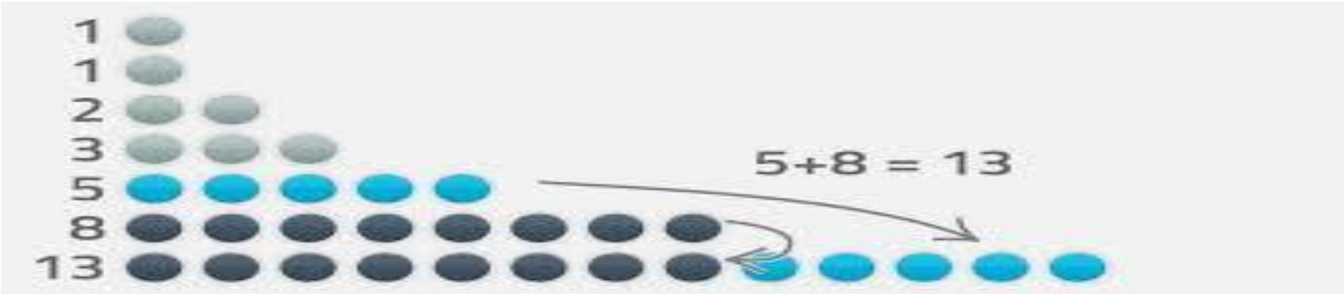
and so we can say that Fibonacci polynomials are defined by second order recurrence relation. So second order recurrence relation is base of the Fibonacci numbers and Fibonacci polynomials.

Fibonacci numbers



Some special types polynomials defined by recurrence relation

Polynomial	Initial value $G_0(x) = p_0(x)$	Initial value $G_1(x) = p_1(x)$	Recursive Formula $G_n(x) = d(x)G_{n-1}(x) + g(x)G_{n-2}(x)$
Fibonacci	1	x	$F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$
Lucas	2	x	$D_n(x) = xD_{n-1}(x) + D_{n-2}(x)$
Pell	1	$2x$	$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x)$
Pell-Lucas	2	$2x$	$Q_n(x) = 2xQ_{n-1}(x) + Q_{n-2}(x)$
Fermat	1	$3x$	$\Phi_n(x) = 3x\Phi_{n-1}(x) - 2\Phi_{n-2}(x)$
Fermat-Lucas	2	$3x$	$\vartheta_n(x) = 3x\vartheta_{n-1}(x) - 2\vartheta_{n-2}(x)$
Chebyshev first kind	1	x	$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$
Chebyshev second kind	1	$2x$	$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$
Jacobsthal	1	1	$J_n(x) = J_{n-1}(x) + 2xJ_{n-2}(x)$
Jacobsthal-Lucas	2	1	$j_n(x) = j_{n-1}(x) + 2xj_{n-2}(x)$
Morgan-Voyce	1	$x+2$	$B_n(x) = (x+2)B_{n-1}(x) - B_{n-2}(x)$
Morgan-Voyce	2	$x+2$	$C_n(x) = (x+2)C_{n-1}(x) - C_{n-2}(x)$

Theorem on recurrence relation sequence

Theorem 1: - Let $c_1, and\ \ c_2$ are arbitrary real numbers and suppose the equation

$$x^2 - c_1x - c_2 = 0 \qquad (1)$$

has $x_1, and\ x_2$ are distinct roots. Then the sequence $< a_n >$ is a solution of the recurrence relation

$$a_n = c_1a_{n-1} + c_2a_{n-2} \quad n \geq 2 \qquad (2)$$

iff

$$a_n = \beta_1x_1^n + \beta_2x_2^n$$

for n= 0, 1, 2, ………………where β_1, and β_2 are arbitrary constants.

Proof: - First suppose that $< a_n >$ of type $a_n = \beta_1x_1^n + \beta_2x_2^n$ we shall prove $< a_n >$ is a solution of recurrence relation (2). Since $x_1, and\ x_2$ roots of equation (1) so all are satisfied equation (1) so we have

$$x_1^2 = c_1x_1 + c_2$$

$$x_2^2 = c_1x_2 + c_2$$

Consider $c_1a_{n-1} + c_2a_{n-2} = c_1(\beta_1x_1^{n-1} + \beta_2x_2^{n-1}) + c_2(\beta_1x_1^{n-2} + \beta_2x_2^{n-2})$

$$= \beta_1x_1^{n-2}(c_1x_1 + c_2) + \beta_2x_2^{n-2}(c_1x_2 + c_2)$$

January - February 2020
ISSN: 0193 - 4120 Page No. 5112 - 5115

$$= \beta_1 x_1^n + \beta_2 x_2^n + \beta_3 x_3^n = a_n$$

This implies

$$c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} = a_n$$

So the sequence $< a_n >$ is a solution of the recurrence relation.

Now we will prove the second part of theorem

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ $n \geq 2$ is a sequence with $initial\ terms$ $a_0 = A_1, a_1 = A_2$

Let $a_n = \beta_1 x_1^n + \beta_2 x_2^n$

So $\beta_1 + \beta_2 = A_1$

(3) $\beta_1 x_1 + \beta_2 x_2 = A_2$

(4)

from (3) we have $\beta_1 = A_1 - \beta_2$, putting this value in (4) we have

$$\beta_2 = \frac{A_1 x_1 - x_2}{x_1 - x_2} \quad \text{and} \quad \beta_1 = \frac{A_2 - A_1 x_2}{x_1 - x_2}$$

Fibonacci polynomials

For any integer $n \geq 0$, the famous Fibonacci polynomials F_n(x) is defined as follows:

$F_0(x)$ = 0, $F_1(x)$ = 1, and $F_{n+1}(x) = F_n(x)$ $+F_{n-1}(x)$ for all $n \geq 1$;

So we have

$$F_2 = x$$
$$F_3 = x^2 + 1$$
$$F_4 = x^3 + 2x$$

So on we can find the all Fibonacci polynomials

Main result of paper

Let For any integer $n \geq 0$, the famous Fibonacci polynomials F_n(x) is defined as follows:

$F_0(x)$ = 0, $F_1(x)$ = 1, and $F_{n+1}(x) = F_n(x)$ $+F_{n-1}(x)$ for all $n \geq 1$; such that $x^2 > (-4)$ then we can written

$$F_n(x) = \frac{1}{2^n\sqrt{x^2+4}}[x + \sqrt{x^2+1}]^n - \frac{1}{2^n\sqrt{x^2+4}}[x - \sqrt{x^2-1}]^n$$

Proof: - using above theorem (1) with $c_1 = x$, $c_2 = 1$ and using T as variable then we have polynomial of above theorem (1)

$$T^2 - xT - 1 = 0$$

and roots of this polynomials are

$$\frac{x+\sqrt{x^2+4}}{2} \text{ and } \frac{x-\sqrt{x^2+4}}{2}$$

By above theorem we have

$$T_n(x) = \beta_1[\frac{x+\sqrt{x^2+4}}{2}]^n + \beta_2[\frac{x-\sqrt{x^2+4}}{2}]^n$$

Using initial terms we have

$$\beta_1 + \beta_2 = 0$$

(a)

$$\beta_1[\frac{x+\sqrt{x^2+4}}{2}]^1 + \beta_2[\frac{x-\sqrt{x^2+4}}{2}]^1 = 1$$

(b)

Solving (a), (b) we have $\beta_1 = \frac{1}{\sqrt{x^2+4}}$, $\beta_2 = \frac{-1}{\sqrt{x^2+4}}$

so we can write

$$F_n(x) = \frac{1}{2^n\sqrt{x^2+4}}[x + \sqrt{x^2+1}]^n - \frac{1}{2^n\sqrt{x^2+4}}[x - \sqrt{x^2-1}]^n$$

Conclusion

Recurrence relation is very useful topic of mathematics many problems of real life many be solved by recurrence relations but in recurrence relation there is a major difficulty in the recurrence relation if we want find 100th term of sequence then we need to find all previous 99 terms of given sequence then we can get 100th term of sequence but above theorem is very useful if coefficients of recurrence relation of given sequence are satisfied the condition of the above theorem then we can apply above theorem and we can find direct value of any term of Fibonacci polynomials sequence at any value of variable x greater than -4.

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January - February 2020
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