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Fractional Time Dependent Model of The Transportation of Heavy Pollutants

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Abstract:
In this paper, investigation of fractional time dependent partial differential equation which acts as governing equation of transportation of dust into atmosphere. In this paper we use Laplace and Hankel transforms in solving fractional partial differential equation. In order to express pollutant concentration, we use confluent hyper-geometric functions. The solution is expressed by using Wright function involving Brownwich representation.

Keywords:*Fractional time dependent partial differential equation, Hankel transforms, Wright function, Brownwich representation, Heavy side function.*

I. Introduction

Consider the governing equation of time dependency partial differential equation:

$$u\frac{\partial q}{\partial x} + w\frac{\partial q}{\partial z} = \frac{\partial}{\partial y}\left(k_y\frac{\partial q}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z\frac{\partial q}{\partial z}\right) \quad (1)$$

Where $q = q$ *(x,y,)* denote the atmospheric concentration of pollutants as defined in [1],

u is denoted as the speed of wind, w is defined as settling velocity of pollutant particles, k_y , k_z are the components of turbulent exchange coefficient. For ground strip $0 \leq x \leq L$ & $-\infty < y < \infty$,

after neglecting y dependency, plane parallel flow of equation is obtained.

$$u\frac{\partial q}{\partial x} + w\frac{\partial q}{\partial z} = \frac{\partial}{\partial z}\left(k_z\frac{\partial q}{\partial z}\right) \quad (2)$$

as in [2] k_z is constant as a linear function of z in [3].

Put $k_z = k_1 z$ and $u = u_1 z^{\mu}$ for some $\mu > 0$

As problem is discussed in [1] but we take care of the fractional order here so our equation reduces to

$$k_1 z\frac{\partial^2 q}{\partial z^2} + (k_1 + w)\frac{\partial q}{\partial z} = u_1 z^{\mu}\frac{\partial q}{\partial x} + \tau_1 z^{\mu}\frac{\partial^{\alpha} q}{\partial t^{\alpha}} \quad (3)$$

Where $0 < \alpha < 1$, and $\tau_1 z^{\mu} = \tau$ is dimensionless parameter.

The auxiliary equation satisfies inside strip $0 \leq x \leq L$, with following conditions

q(0,z,t) = *q(x,z,t)* = *0*&$\lim_{z\to\infty} q(x,z,t) = 0$

$-[k_1 z q_z(x,y,z) + wq(x,z,t)]_{z=0} = f(x,t)$ and

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$\lim_{z\to\infty} z^{\gamma}\, q_z(x,z,t) = 0$, $\gamma = \frac{3-\mu}{4} + w/2k1$,

Outside strip it is assumed that

$k_1 z q_z(x,y,z)|_{z=0} = 0.$

The general form of Hankel transform is defined as

$$\mathcal{H}_v\,[g(z); s,a,c] = \int_0^{\infty} z^a\, g(z) J_v(sz^c) dz = G_v(s)$$

$g(z)$ is defined as a function for which integral should exists. In order to obtain inversion formula we have

$$g(z) = \mathcal{H}_v{}^{-1}[G(z); s,a,c] = cz^{2c-a-1}\int_0^{\infty} s\, G_v(s) J_v(sz^c) ds \quad (c > 0)$$

The properties of these transforms are defined in [3]. Here we define another operator

$$Q = \wedge_z \frac{d^2}{dz^2} + \rho\, \frac{d}{dz} - \lambda(c^2v^2 - a^2)z^{-1} \quad ,$$

and here we found that

$$\mathcal{H}_v\,[Qg(z); s,a,c] = \wedge - \lambda c^2 z^2 \mathcal{H}_v\,[z^{2c-1} g(z); s,a,c]$$

where

$$\wedge = [\, z^a \left(\lambda z \frac{dg\,(z)}{dz} + \rho g(z)\right) J_v(sz^c)]_0^{\infty} - [\lambda g(z\,) \frac{d}{dz} z^{a+1} J_v(sz^c)]_0^{\infty}$$

Constant parameters satisfy relation $\frac{\rho}{\lambda} = 2a+1$. Now Laplace transform of function is $g(t\,)$ is given by

$$\mathrm{L}\quad g(t\,) = \bar{g}(p) = \int_0^{\infty} e^{-pt}\, g(t)dt \quad , \quad p > 0$$

By making different choices of *f(x,t)* we obtain solution of *q(x,z,t)* in terms of confluent hyper-geometric function of second kind given as

$$\varphi(a,b,z) = \frac{1}{\Gamma(a)} \int_0^{\infty}\int_0^{\infty} e^{-zt} t^{a-1}\, (1t)^{b-a-1} dt,$$
$$Re(a), Re(b) > 0.$$

Moreover we define Heavy side function $\dot{\mathrm{H}}$:

$$\begin{cases} \dot{\mathrm{H}}\,(\mathrm{t}) = 1 & \mathrm{t} > 0 \\ \dot{\mathrm{H}}\,(\mathrm{t}) = 0 & \mathrm{t} < 0 \end{cases}$$

which is given as time factor having the representation of *q(x,z,t).*

II. Solution of the problem

The solution of the problem in eq. 3 is transformed into fractional partial differential equation in (x,t).

$$\wedge - \lambda c^2 s^2 \mathcal{H}_v\,[z^{\mu} q(x,z,t); s,a,c] = [u_1 \frac{\partial}{\partial x} + \tau_1 \frac{\partial^{\alpha}}{\partial t^{\alpha}}] \mathcal{H}_v\,[z^{\mu} q(x,z,t); s,a,c] \qquad (4)$$

In which parameters are defined as $\lambda = k_1$,

$$\rho = k_1 + w\,, \qquad v = -\frac{w}{k_1(1+\mu)},$$

$a = -\frac{v(1+\mu)}{2}$, $c = -\frac{a}{v}$ from initial boundary condition on q we have

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$$\wedge = \frac{s^v f(x,t)}{2^v \Gamma(v+1)}$$

Applying Laplace transform to eq. 4 we get,

$$u_1 \frac{\partial}{\partial x} \mathcal{H}_v \left[z^\mu \bar{q}(x,z,p); s,a,c\right]$$

$+$
$(\lambda c^2 s^2 \ + \tau_1 p^\alpha) \mathcal{H}_v \left[z^\mu \bar{q}(x,z,p); s,a,c\right]$

$= \frac{s^v \bar{f}(x,t)}{2^v \Gamma(v+1)}$ Whose solution is given by

$$\mathcal{H}_v \left[z^\mu \bar{q}(x,z,p); s,a,c\right] = \frac{s^v e^{-(\lambda c^2 s^2 + \tau_1 p^\alpha) x/u_1}}{u_1 2^v \Gamma(v+1)} \int_0^x e^{\frac{(\lambda c^2 s^2 + \tau_1 p^\alpha) r}{u_1}} \bar{f}(r,p) dr$$

Apply Hankel transform, we get

$$\bar{q}(x,z,p) = \frac{c z^{-\mu+2c-a-1}}{u_1 2^v \Gamma(v+1)} \int_{[0}^{\infty} \bar{f}(r,p) \left[\int_0^\infty s^{v+1} J_v(s z^c)\, e^{-(\lambda c^2 s^2 + \tau_1 \underline{p}^\alpha) x - \frac{r}{u_1}} ds\right] dr$$

To evaluate the S- integral we make the use of Weber's integral formula as described in [4] by changing a, c, λ and change the variable of integration, gives

$$\bar{q}(x,z,p) = \frac{(u_1/k_1)^{v+1} z^{v(1+\mu)}}{(1+\mu)^{2v+1} u_1 \Gamma(v+1)} \int_0^x \frac{\bar{f}(x-r,p)}{r^{v+1}}$$

$$\text{X} \ \exp\left[-\frac{u_1 z^{\mu+1}}{k_1 (1+\mu)^2 r} - \frac{\tau_1 p^\alpha r}{u_1}\right] dr \qquad (5)$$

Taking inverse Laplace of eq. 5 and using $K(z) = \frac{u_1 z^{\mu+1}}{k_1 (1+\mu)^2}$

$$q(x,z,t) = \frac{[K(z)]^v}{(1+\mu)\Gamma(v+1)} \int_0^x \frac{\bar{f}(x-r,p)}{r^{v+1}}$$

$$\text{X} \qquad \exp\left[-\frac{K(z)}{r} - \frac{\tau_1 p^\alpha r}{u_1}\right] dr$$

(6)

Here we make use of

$$f(x,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\varphi_{m,n} x^m t^m}{m!\, n!} = 1$$

$$\bar{f}(x,p) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\varphi_{m,n} x^m}{m!\, n!} \frac{\Gamma(n+1)}{\rho^{n+1}}$$

Now $\quad L^{-1}\left[\bar{f}(x-r,p).\, e^{-\frac{\tau_1 p^\alpha r}{u_1}}\right] =$
$L^{-1}\left[\sum_{m,n=0}^{\infty} \frac{\varphi_{m,n} (x-r)^m}{m!n!} \frac{\Gamma(n+1)}{\rho^{n+1}} e^{-\frac{\tau_1 p^\alpha r}{u_1}}\right]$

$$\sum_{m,n=0}^{\infty} \frac{\varphi_{m,n} (x-r)^m}{m!} \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{e^{pt} e^{-\frac{\tau_1 p^\alpha}{u_1}}}{p^{n-1}} dp$$

Using $pt = \sigma$ we get $dp = \frac{d\sigma}{t}$

$=$

$$\sum_{m,n=0}^{\infty} \frac{\varphi_{m,n} (x-r)^m}{m!} \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{t^{n+1} e^{\sigma - \frac{\tau_1 r \sigma^\alpha}{\mu\ t^\alpha}}}{\sigma^{n+1}} \frac{d\sigma}{t}$$

Using definition of Wright function

$= \sum_{m,n=0}^{\infty} \frac{\varphi_{m,n} (x-r)^m}{m!} t^n\, W\left(\frac{\tau_1 r}{\mu_1 t^\alpha}, -\alpha, n + 1\right); -\alpha > -1,\ \alpha < 1,\ n+1 > 0$

Using $K(z) = \frac{\mu_1 z^{1+\mu}}{k_1 (1+u)^2}$, we get from eq. 6

$$q(x,z,t) = \frac{[K(z)]^{v}}{(1+\mu)k_1\sigma(r+1)}\int_0^{\infty}\frac{e^{-\frac{K(z)}{r}}}{r^{v+1}}\sum_{m,n=0}^{\infty}\frac{\varphi_{m,n}(x-r)^m}{m!}t^n$$

$$.\ W\left(\frac{\tau_1 r}{\mu_1 t^{\alpha}},-\alpha,n+1\right)$$

III. Conclusion

The solution obtained here in the form of Wright function having Brownwich representation. The solution obtained here is much more convenient as before it was obtained for non-fractional order partial differential equation. If we give value $\alpha = 1$ the solution obtained will be same as in [1].

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