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Algorithms for Calculating Eigen Values of Differential Operators with Unsmooth Potential on A Projective Plane

Olga Torshina[1*], Elena Moskvina[1], Yuliya Somova[1], Mars Khayrullin[2], Elizabeth Dick[3], Svetlana Arslanbekova[3], Hodak Sergey[2], Andrey Goncharov[2], Maria Belousova[2]

[1]Nosov Magnitogorsk State Technical University, Magnitogorsk, Russian Federation
[2]K.G. Razumovsky Moscow State University of technologies and management (the First Cossack University), Moscow, Russian Federation
[3]Bashkir State Agrarian University, Ufa, Russia
*Corresponding author: rebezov@ya.ru

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Abstract:
The modeling of various processes in natural and engineering sciences in some cases leads to problems of finding the eigenvalues of operators. The problems of the hydrodynamic theory of stability, electric oscillations in a long line, seismic prospecting, problems of non-destructive testing, image processing, composite materials identification, etc. are a good example. The solving of a wide range of natural science problems which involves finding eigenvalues of differential operators with complex spectral parameter, as a rule, means finding asymptotic formulas. The latter, in turn, seriously complicates the process of obtaining the desired result. The addition theorem allows to circumvent these difficulties in the case of considering operators with a unsmooth potential on the projective plane. On its basis, some fairly effective algorithms for calculating perturbation theory corrections have been developed.

Keywords:mathematical modeling, differential operators, eigenvalues, projective plane, unsmooth potential

1. Introduction

The spectral theory of differential operators is the most important part of the general spectral theory of operators and occupies a prominent place in the mathematical research of the 19th and 20th centuries, as well as in numerous applications of mathematics to various physics theories.

The spectral theory of differential operators originates in the theory of eigenvalues and eigenfunctions of boundary value problems of mathematical physics. The origin of the latter refers to the XVIII century and is associated with the works of D. Bernoulli, L. Euler and J. D'Alembert on string oscillations. Mainly, the interest arose in trigonometric systems of functions. The initial stage of the research development in this area ends with the work of J. Fourier.

The start of the general theory of boundary value problems related to a differential equation of the second order was made in 1830 with the J. Sturm

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and J. Liouville's works. The most important result of these studies was the proof of the existence of a sequence of eigenvalues and a sequence of eigenfunctions of the boundary value problem. At the same time, scientists considered the possibility of expanding a function belonging to a certain class in a series in a system of eigenfunctions.

In the second half of the XIX century, starting with the P.L. Chebyshev's works, the study of systems of orthogonal functions became independent. In subsequent studies, considerable attention is paid to the systems of orthogonal polynomials. These include the studies of E. Laguerre, S. Hermite, I. Gram, N. Y. Sonin and others. The theory of the expansion of functions into series in systems of orthogonal functions arising when solving boundary problems of mathematical physics is significantly developed in the works of V.A. Steklov. For its further development, a great contribution was made by the Gauss's method of least squares and the research of P.L Chebyshev establishing a connection between the theory of the expansion of functions into series of orthogonal functions and the problem of the best quadratic approximation of functions.

New creative ideas in the theory of eigenvalues and eigenfunctions of mathematical physics appeared at the end of the 19th century due to the development of the theory of linear integral equations. Especially significant results in the theory of linear integral equations were obtained by I. Fredholm. The scientist was guided by the analogy of a linear integral equation with a system of linear algebraic equations. A new stage in the development of the theory of eigenvalues is associated with the name of D. Hilbert. Fundamental research of a scientist on the general theory of linear integral equations (1904-1910) led to the introduction of one of the basic mathematical concepts of the 20th century - the Hilbert space.

The integration of geometric ideas and images with the abstract concepts of the set theory on the basis of relevant analytical theories was very productive. The classical theory of eigenvalues of boundary value problems for differential equations, namely, the Sturm-Liouville problem allowed as 4 factors of the differential expressions only continuous function on a finite closed interval. The theory of symmetric linear integral equations contributed to a significant extension of the theory of eigenvalues and eigenfunctions for new classes of differential equations and boundary value problems. Under the influence of D. Hilbert's ideas, a great number of studies related to the problem of the expansion in eigenfunctions of differential equations of second and higher order were performed. Among them, there is the works of E. Schmidt, A. Miller, B. L. Bunitsky, A. Kneser, M. Plancherel, J. Tamarkin, E. Hilbe and others.

A new important move in the direction of the study of differential operators was made thanks to the spectral theory of symmetric limited and particular cases of unbounded bilinear forms developed by D. Hilbert. Thus, in the works of G. Weil (1908–1910), the theory of expansion in eigenfunctions of second-order differential operators for singular cases was first stated. Thus, the beginning of the general spectral theory of ordinary singular differential operators was made. In the works of D. Hilbert and his followers, in particular, G. Weil, the classical mathematical apparatus is mainly used. At the same time, the Hilbert space theory started its developing in an abstract form. The development of the theory of concrete Hilbert spaces — the space of sequences and the space of functions with an integrable square was of great importance for the formation of a general functional analysis. For two decades (1910–1930), the theory of linear operators in Hilbert space had been taken a completely modern form. The central place in this theory is occupied by the spectral theory. An axiomatic presentation of the theory of linear operators in a Hilbert space

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was given almost simultaneously in the works of J. von Neumann (1929), M. Stone (1929) and in the book (1932) by F. Riss (1930). Later presentations of the theory of linear operators in a Hilbert space, along with methodological improvements, contained the results of new research. A fairly complete description of the modern state of the theory of linear operators in a Hilbert space can be given by the book of N.I. Akhiezer and I.M. Glasman "The theory of linear operators in Hilbert space", N. Dunford and J.T. Schwarz "Linear operators", A.I. Plesner "The Spectral Theory of Linear Operators", F. Riss and B. Szőkefalvi-Nagy "Lectures on Functional Analysis", K. Morin "Methods of Hilbert Space". By the mid-20s of the 20th century, the results which were obtained in the theory of Hilbert space, especially in the spectral theory of linear operators, turned out to be so complete and perfect that the new mathematical theory was able to successfully respond to queries of the rapidly developing physical quantum theory. The spectral theory of linear operators in Hilbert space was the mathematical basis of E. Heisenberg's "matrix mechanics" and E. Schrödinger's "wave mechanics." The application of the spectral theory of linear operators in quantum physics gave new benchmarks and opportunities for further development of the spectral theory. A fundamental statement of the mathematical apparatus in quantum mechanics was presented in 1932 by J. von Neumann in the book "Mathematical Foundations of Quantum Mechanics".

In the following years, special attention is paid to the study of singular differential operators simultaneously with the development of the general spectral theory of linear operators and its application to the study of specific operators. The methods of the general spectral theory of linear operators in a Hilbert space often turned out to be insufficiently flexible in some aspects of the theory of singular differential operators. For instance, the direct application of analytical methods turned out to be more efficient to study the properties of the spectrum of differential operators depending on the behavior of the factors of the operators. The spectral theory of differential operators is presented without formal involvement of the general theory of linear operators in a Hilbert space in the monograph by E.Ch. Titchmarsh "Expansions in eigenfunctions associated with second-order differential equations" (1946). Also, the works of M. G. Krein on the theory of extensions of symmetric operators and the method of directing functionals are important in the development of the spectral theory of singular differential operators. Using the method mentioned above M.G. Kerin proved the eigenfunction expansion theorem for ordinary self-adjoint differential operators. The M.G. Kerin method of guiding functionals is the connecting link between the general theory of linear operators and the theory of expansion in eigenfunctions of differential operators. Another proof of the expansion theorem for ordinary differential operators was given in 1949-50 in the works of K. Kodaira. Later B.M. Levitan, K. Yosida and N. Levinson obtained new proofs of the decomposition theorem.

The next stage in the development of the spectral theory of differential operators is associated with its formation as an independent section of functional analysis with its own tasks and methods. This is due to the advent of monographs by E.Ch. Titchmarsh, B.M. Levitan (1950) and M.A. Naimark (1954). So, in the book of M. A. Naimark, the theory of differential operators is presented for operators of arbitrary order. A great number of research papers are related to the study of the central problem of the spectral theory of differential operators, as mentioned above, the theorem on expansion in eigenfunctions. In addition to the proof of the main decomposition theorem, there arises the question about the uniqueness of the decomposition or the uniqueness of the so-called spectral function. Another equally important problem of the spectral

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theory of differential operators is the problem of expansion of operators. A large number of studies have been devoted to the study of the asymptotic properties of eigenvalues, the spectrum, and eigenfunctions. The main tasks of the spectral theory include determining the defect index of a differential operator depending on the behavior of the coefficients of the differential expression. The most important task of the spectral theory of differential operators is the characteristic of the spectrum of the operator, also depending on the behavior of the coefficients of the differential expression generating the operator. The results obtained in this field give an incomplete idea of the nature of the spectrum. Individual classes of operators with a discrete spectrum have been studied, and some information has been obtained about the location of the continuous part of the spectrum. Direct methods for the qualitative spectral analysis of singular differential operators are being successfully developed. An overview of the results got by these methods is presented in the monograph by I.M. Glasman.

The spectral theory of higher-order differential operators is comparatively little studied. The complete solution of the inverse 6 problem of spectral analysis related to the definition of an ordinary differential operator by the spectral function is given in the works of I.M. Gelfand, B.M. Levitan, M. G. Kerin, V.A. Marchenko. A considerable circle of problems were posed in connection with the problem of constructing a spectral theory of non-self-adjoint differential operators. The new direction of the spectral theory of differential operators gets in the transition to the space of generalized functions. The mathematical analysis of physical problems continues to serve as a beneficial source for the development of the spectral theory of differential operators.

The areas of application of the theory of differential operators are constantly expanding. So, in particular, the study of scientists W.N. Traves [28], T. Levasseur and J. Stafford [29], Schwartz [30].]. Constructive invariant theory was a preoccupation of many nineteenth century mathematicians, but the topic fell out of fashion in the early twentieth century. In the latter twentieth century the topic enjoyed resurgence, partly due to its connections with the construction of moduli spaces in algebraic geometry and partly due to the development of computational algorithms suitable for implementation in modern symbolic computation packages.

Nowadays, the ideas of an effective method for the approximate calculation of eigenvalues and eigenfunctions of perturbed self-adjoint operators, called the method of regularized traces by the authors, were formulated in the works of V.A. Sadovnichy, V.V. Dubrovsky and S.I. Kadchenko. A feature of the method is the fact that it is based not on the matrix representation of discrete operators, but on the spectral characteristics of the unperturbed operator and the spectrum of the perturbed operator. Besides, while developing the Galerkin method, some linear formulas are obtained for calculating the approximate eigenvalues of discrete lower semibounded operators. The formulas allow one to calculate the eigenvalues of the specified operators of any number, regardless of whether the eigenvalues with the preceding numbers are known or not. At the same time, it is possible to calculate eigenvalues with large numbers, when the application of the Galerkin method becomes difficult. A numerical implementation is also possible, for example, in the work [13] there is a well-known algorithm for solving the complete eigenvalue problem and also its implementation in the Fortran language is given. Under the proposed article, we are going to consider the calculation of eigenvalues for operators with unsmooth potential on the projective plane.

Let

$$T = -\Delta = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)$$

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the standard Laplace-Beltrami operator with a potential on the projective plane F, acting in the Hilbert space H of functions that are square-integrable according to the Haar measure: $\sin\theta d\varphi d\theta$ (θ,φ -spherical coordinates), $\lambda_n = n(n+1)$ $\left(n=\overline{0,\infty}\right)$are the eigenvalues of the operator T, $\nu_n = 2n+1$is the multiplicity of the eigenvalue λ_n; $v_{n,i}$ $\left(i=\overline{0,2n}\right)$ is eigenfunctions of the operator T, forming a system of orthonormal spherical functions. Let also $l_n = \{\lambda \mid \lambda = \lambda_n + n + 1 + ip, -\infty < p < +\infty\}$be vertical lines on the complex plane. Let denote with $\mu_{n,i}$ the eigenvalues of the operator $T+P$ taken with regard to algebraic multiplicity, such that
$\left|\mu_{n,i} - n(n+1)\right| \le const$.

2. Research methods

When solving this problem, we used methods of functional analysis, spectral analysis of linear operators, and perturbation theory. Also, in the work, new methods developed by the Moscow Scientific School by V.A. Sadovnichiy, the member of Russian Academy of Science.

Let us consider a spherical order function n

$$P_n(\cos\gamma) = 2\sum_{m=0}^{\infty}\frac{(n-m)!}{(n+m)!\,\delta_m}P_n^m(\cos\theta')P_n^m(\cos\theta)\cos m(\varphi-\varphi'),$$

where γ is the angle between the radius vectors in spherical coordinates $(\theta,\varphi)$and (θ',φ');

$\cos\gamma = \sin\theta\sin\theta'\cos(\varphi-\varphi') + \cos\theta\cos\theta'$;

$\delta_m = 2$ when $m=0$ and $\delta_m = 1$ when $m>0$;

$$P_n^m(x) = \left(1-x^2\right)^{m/2}\frac{d^m P_n(x)}{dx^m} = \frac{\left(1-x^2\right)^{m/2}}{2^n n!}\frac{d^{n+m}}{dx^{n+m}}\left[\left(x^2-1\right)^n\right]$$

;

polynomial

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[\left(x^2-1\right)^n\right],\ -1<x<1.$$

Obviously equality is true.

$$\sum_{i=0}^{2n}\mu_{n,i} = n(n+1)(2n+1) + \sum_{i=0}^{2n}(Pv_{ni}, v_{ni}) + \alpha_n(p) + \beta_n(p) + O\left(\frac{1}{n^2}\right),$$

where $\alpha_n(p)$is the second amendment of the perturbation theory to the sum $\sum_{i=0}^{2n}\mu_{n,i}$, and $\beta_n(p)$ is the third amendment of the perturbation theory.

The first amendment is equal to the constant

$$\sum_{i=0}^{2n}(Pv_{ni}, v_{ni}) = \frac{2m+1}{4\pi}\iint_{\Phi} p(\theta,\varphi)\sin\theta d\varphi d\theta = const$$

The second amendment of the perturbation theory is

$$\alpha_n(p) = -\frac{1}{2\pi i}Sp\left\{\left[\int_{l_n} - \int_{l_{n-1}}\right]\lambda\left[(T-\lambda E)^{-1}P^2\right](T-\lambda E)^{-1}d\lambda\right\} = -\sum_{k=1}^{\infty}\sum_{k\ne n}\frac{\alpha_{k,n}}{\lambda_k - \lambda_n}$$

.

By the addition theorem for even spherical harmonics, we have

$$\alpha_{k,n} = \sum_{i=0}^{2n}\sum_{j=0}^{2k}(Pv_{ki}, v_{nj})(Pv_{nj}, v_{ki}) =$$

$$= \iint_{\Phi}\iint_{\Phi}\frac{(2k+1)(2n+1)}{4\pi^2}p(\theta,\varphi)p(\theta',\varphi')P_k(\cos\alpha)P_n(\cos\alpha)L_i\sin\theta L_j\sin\theta' d\theta d\varphi d\theta' d\varphi'$$

,

where

$\cos\alpha = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')$, the Legendre polynomials P_k, P_n are normalized by the condition $P_k(1) = P_n(1) = 1$.

Let us introduce a function

$$f(\alpha) = \iint_{\Phi} p(\theta,\varphi)L_j\sin\theta\, d\theta d\varphi\left[\iint_{T(\alpha)} p(\theta',\varphi)' L_j'\sin\theta'\phi(\alpha,\theta,\theta')d\theta'\right]$$

,

where $T(\alpha)$is the intersection of a cone having a vertex at the center of the sphere, a central angle $2\alpha\,(0\le\alpha\le\pi)$and an axis defined by spherical coordinates θ,φ, with a sphere in spherical coordinates θ',φ';

$$\phi(\alpha,\theta,\theta') = \left(\sin^2\theta\sin^2\theta' - \cos^2\alpha - \cos^2\theta\cos^2\theta' + 2\cos\alpha\cos\theta\cos\theta'\right)^{-1/2}$$

;

$$L_j = L_j\left(\frac{d}{d\theta}\right),\ L_j' = L_j\left(\frac{d}{d\theta'}\right),\ (j=1,2).$$

According to the condition of the theorem, the function p satisfies the Lipchitz condition.

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Consequently, the function f also satisfies the Lipchitz condition, namely:

$$|f(\alpha)-f(\beta)|=\left|\iint_{\Phi} p(\theta,\varphi)L_j\sin\theta\, d\theta d\varphi\left[\iint_{T(\alpha)} p(\theta',\varphi)' L_j'\sin\theta'\phi(\alpha,\theta,\theta')d\theta' -\right.\right.$$

$$\left.\left. -\iint_{T(\alpha)} p(\theta',\varphi)' L_j'\sin\theta'\phi(\alpha,\theta,\theta')d\theta'\right]\right| \le const|\alpha-\beta|.$$

This fact allows us to conclude that the function f is absolutely continuous. Therefore, at each point $[0,\pi]$ it has a finite derivative $f'(\alpha)$ which turns out to be a summarized function.

Based on the above, the second amendment of the perturbation theory is (we will choose $\varepsilon>0$ later)

$$\alpha_n(p)=-\sum_{k=1,k\ne n}^{\infty}\frac{(2k+1)(2n+1)}{4\pi^2}\frac{1}{|\lambda_k-\lambda_n|}\int_0^{\pi} f(\alpha)\sin\alpha P_k(\cos\alpha)P_n(\cos\alpha)d\alpha=$$

$$=-\sum_{k=1,k\ne n}^{\infty}\frac{(2k+1)(2n+1)}{4\pi^2}\frac{1}{|\lambda_k-\lambda_n|}\left\{\left[\int_0^{\varepsilon}+\int_{\pi-\varepsilon}^{\pi}\right]f(\alpha)\sin\alpha P_k(\cos\alpha)P_n(\cos\alpha)d\alpha+\right.$$

$$\left.+\int_{\varepsilon}^{\pi-\varepsilon} f(\alpha)\sin\alpha P_k(\cos\alpha)P_n(\cos\alpha)d\alpha\right\}.$$

For the Legendre polynomials, the asymptotic Stieltjes decomposition with a uniform estimate of the remainder is known [1].

$$P_k(\cos\alpha)=\frac{\cos\{(k+1/2)\alpha-\pi/4\}}{(\sin\alpha)^{1/2}}\left[\frac{(2/\pi)^{1/2}}{k^{1/2}}+\frac{O(1)}{k^{3/2}}\right]+$$

$$+\frac{\sin\{(k+3/2)\alpha-\pi/4\}}{(\sin\alpha)^{3/2}}\frac{O(1)}{k^{3/2}}+\frac{O(1)}{(\sin\alpha)^{5/2}k^{5/2}}\quad .$$

Using this asymptotic expansion, we transform $\alpha_n(p)$ in the following way:

$$\alpha_n(p)=-\sum_{k=1,k\ne n}^{\infty}\frac{(2k+1)(2n+1)}{4\pi^2|\lambda_k-\lambda_n|}\left\{\left[\int_0^{\varepsilon}+\int_{\pi-\varepsilon}^{\pi}\right]f(\alpha)\sin\alpha P_k(\cos\alpha)P_n(\cos\alpha)d\alpha+\right.$$

$+$

$$\int_{\varepsilon}^{\pi-\varepsilon} f(\alpha)\cos\{(k+1/2)\alpha-\pi/4\}\cos\{(n+1/2)\alpha-\pi/4\}\times$$

$$\int_{\varepsilon}^{\pi-\varepsilon} f(\alpha)\cos\{(k+1/2)\alpha-\pi/4\}\cos\{(n+1/2)\alpha-\pi/4\}\times+$$

$+$

$$\int_{\varepsilon}^{\pi-\varepsilon}\frac{f(\alpha)}{\sin\alpha}\cos\{(k+1/2)\alpha-\pi/4\}\sin\{(n+3/2)\alpha-\pi/4\}\left\{\frac{O(1)}{k^{1/2}n^{3/2}}+\frac{O(1)}{k^{3/2}n^{3/2}}\right\}d\alpha$$

$+$

$$+\int_{\varepsilon}^{\pi-\varepsilon}\frac{f(\alpha)}{\sin^2\alpha}\cos\{(k+1/2)\alpha-\pi/4\}\frac{O(1)}{k^{1/2}n^{5/2}}d\alpha+$$

$+$

$$\int_{\varepsilon}^{\pi-\varepsilon}\frac{f(\alpha)}{\sin\alpha}\sin\{(k+3/2)\alpha-\pi/4\}\cos\{(n+1/2)\alpha-\pi/4\}\frac{O(1)}{n^{1/2}k^{3/2}}d\alpha+$$

$+$

$$\int_{\varepsilon}^{\pi-\varepsilon}\frac{f(\alpha)}{\sin^2\alpha}\sin\{(k+3/2)\alpha-\pi/4\}\sin\{(n+3/2)\alpha-\pi/4\}\frac{O(1)}{n^{3/2}k^{3/2}}d\alpha+$$

$$+\int_{\varepsilon}^{\pi-\varepsilon}\frac{f(\alpha)}{\sin^3\alpha}\sin\{(k+3/2)\alpha-\pi/4\}\frac{O(1)}{n^{5/2}k^{3/2}}d\alpha+$$

$$+\int_{\varepsilon}^{\pi-\varepsilon}\frac{f(\alpha)}{\sin^2\alpha}\cos\{(k+1/2)\alpha-\pi/4\}\frac{O(1)}{k^{1/2}n^{5/2}}d\alpha+$$

$$+\int_{\varepsilon}^{\pi-\varepsilon}\frac{f(\alpha)}{\sin^3\alpha}\sin\{(n+3/2)\alpha-\pi/4\}\frac{O(1)}{n^{3/2}k^{5/2}}d\alpha+$$

$$\left.+\int_{\varepsilon}^{\pi-\varepsilon}\frac{f(\alpha)}{\sin^4\alpha}\frac{O(1)}{n^{5/2}k^{5/2}}d\alpha\right\}=\sum_{i=0}^{13}a_n^i(p).$$

Let us estimate the items $\alpha_n(p)$.

$$|\alpha_n^0(p)|\le O(1)\sum_{k=1,k\ne n}^{\infty}\frac{k^{1/2}n^{1/2}}{|k-n|(k+n+1)}\left[\int_0^{\varepsilon}+\int_{\pi-\varepsilon}^{\pi}\right]|f(\alpha)|d\alpha=O(\varepsilon)\sum_{k=1,k\ne n}^{\infty}\frac{k^{1/2}n^{1/2}}{|k-n|(k+n+1)}=O(\varepsilon\ln n)$$

In this assessment, we used inequalities [2]:

$$\sqrt{\sin\alpha}|P_k(\cos\alpha)|<\sqrt{\frac{2}{\pi k}},\quad \sqrt{\sin\alpha}|P_n(\cos\alpha)|<\sqrt{\frac{2}{\pi n}}$$

,

as well as the condition that near $\alpha=0$ and $\alpha=\pi$ $|f(\alpha)|=O(\alpha)$.

Further

$$\alpha_n^1(p)=-\frac{1}{4\pi^3}\sum_{k=1,k\ne n}^{\infty}\frac{(2k+1)(2n+1)}{|k-n|(k+n+1)}\int_{\varepsilon}^{\pi-\varepsilon}f(\alpha)[\sin((k+n+1)\alpha)+\cos((k-n)\alpha)]\frac{d\alpha}{k^{1/2}n^{1/2}}=\varphi_1+\varphi_2.$$

Using integration by parts, we obtain the following equalities:

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$$\int_{\varepsilon}^{\pi-\varepsilon} f(\alpha)sin((k+n+1)\alpha)d\alpha = O\left(\frac{\varepsilon}{k+n}\right) + O\left(\frac{1}{(k+n)^2}\right),$$

$$\int_{\varepsilon}^{\pi-\varepsilon} f(\alpha)cos((k-n)\alpha)d\alpha = O\left(\frac{\varepsilon}{k-n}\right) + O\left(\frac{1}{(k-n)^2}\right).$$

In this case:

$$\varphi_1 = O(\varepsilon)\sum_{k=1,k\neq n}^{\infty} \frac{k^{1/2}n^{1/2}}{|k-n|(k+n)^2} = O\left(\frac{ln\,n}{n}\varepsilon\right),$$

let $l = k-n$, then

$$\varphi_2 = -\frac{1}{4\pi^3}\sum_{l=1-n,l\neq 0}^{\infty} \frac{(2n+2|l|+1)(2n+1)}{|l|(2n+|l|+1)n^{1/2}(n+|l|)^{1/2}} \int_{\varepsilon}^{\pi-\varepsilon} f(\alpha)cos(|l|\alpha)d\alpha =$$

$$= -\frac{1}{4\pi^3}\sum_{l=1-n,l\neq 0}^{\infty} \frac{1}{|l|}\frac{(2n)(2n)}{2nn^{1/2}n^{1/2}} \frac{\left(1+(2|l|+1)(2n)^{-1}\right)\left(1+(2n)^{-1}\right)}{\left(1+(|l|+1)(2n)^{-1}\right)\left(1+|l|(n)^{-1}\right)^{1/2}} \int_{\varepsilon}^{\pi-\varepsilon} f(\alpha)cos(|l|\alpha)d\alpha =$$

$$= -\frac{1}{2\pi^3}\left\{\sum_{l=1-n,l\neq 0}^{\infty} \frac{1}{|l|}\int_{\varepsilon}^{\pi-\varepsilon} f(\alpha)cos(|l|\alpha)d\alpha + \frac{1}{2n}\sum_{l=1-n,l\neq 0}^{\infty} \frac{1}{|l|}\int_{\varepsilon}^{\pi-\varepsilon} f(\alpha)cos(|l|\alpha)d\alpha + \right.$$

$$\left. + \sum_{l=1-n,l\neq 0}^{\infty} O\left(\frac{1}{n^2}\right)\int_{\varepsilon}^{\pi-\varepsilon} f(\alpha)cos(|l|\alpha)d\alpha\right\} = O\left(\frac{\varepsilon}{n}\right).$$

As a result, we get

$$\alpha_n^1(p) = O\left(\frac{\varepsilon\, ln\,n}{n}\right) + O\left(\frac{\varepsilon}{n}\right) = O\left(\frac{\varepsilon\, ln\,n}{n}\right).$$

Let us consider

$$\alpha_n^2(p) = -\frac{1}{4\pi^3}\sum_{k=1,k\neq n}^{\infty} \frac{(2k+1)(2n+1)}{|k-n|(k+n+1)} \int_{\varepsilon}^{\pi-\varepsilon} f(\alpha)[sin((k+n+1)\alpha) + cos((k-n)\alpha)]\frac{O(1)}{k^{3/2}n^{1/2}}d\alpha =$$

$$= O(1)\sum_{k=1,k\neq n}^{\infty} \frac{1}{k^{1/2}}\frac{n^{1/2}}{|k^2-n^2|}\left[\int_{\varepsilon}^{\pi-\varepsilon} f(\alpha)sin((k+n+1)\alpha)d\alpha + \int_{\varepsilon}^{\pi-\varepsilon} f(\alpha)cos((k-n)\alpha)d\alpha\right] =$$

$$= O(\varepsilon)\sum_{k=1,k\neq n}^{\infty}\left[\frac{1}{k^{1/2}|k-n|n^{3/2}} + \frac{n^{1/2}}{k^{1/2}(k-n)^2(k+n)}\right] = O\left(\frac{\varepsilon}{n^{3/2}}\right) + O\left(\frac{\varepsilon}{n}\right) + O\left(\frac{\varepsilon\, ln\,n}{n^2}\right)$$

.

Assessing the next item, we get:

$$\alpha_n^3(p) \leq O(\varepsilon)\sum_{k=1,k\neq n}^{\infty}\left[\frac{1}{n|k^2-n^2|} + \frac{1}{n(k-n)^2}\right] = O\left(\frac{\varepsilon\, ln\,n}{n^2}\right) + O\left(\frac{\varepsilon}{n^2}\right).$$

Let us consider

$$\alpha_n^5(p) = -\sum_{k=1,k\neq n}^{\infty} \frac{(2k+1)(2n+1)}{8\pi^2|k-n|(k+n+1)k^{1/2}n^{3/2}} \int_{\varepsilon}^{\pi-\varepsilon} \frac{f(\alpha)}{\sin\alpha}[\sin((k-n-1)\alpha) - \cos((k+n+2)\alpha)]d\alpha = \gamma_1 + \gamma$$

$$\gamma_1 = -\sum_{l=1-n,l\neq 0}^{\infty} \frac{(2n+2|l|+1)(2n+1)}{|l|(2n+2|l|+1)(n+|l|)^{1/2}n^{3/2}} \int_{\varepsilon}^{\pi-\varepsilon} \frac{f(\alpha)}{\sin\alpha}\sin((|l|-1)\alpha)d\alpha =$$

$$= -\frac{1}{4\pi^2 n}\sum_{l=1-n,l\neq 0}^{\infty} \frac{1}{|l|n}\int_{\varepsilon}^{\pi-\varepsilon} \frac{f(\alpha)}{\sin\alpha}\sin((|l|-1)\alpha)d\alpha =$$

$$= -\frac{1}{4\pi^2 n}\sum_{l=1-n,l\neq 0}^{\infty} \frac{1}{|l|n}\int_{\varepsilon}^{\pi-\varepsilon} f(\alpha)ctg\alpha\sin(|l|\alpha)d\alpha + O\left(\frac{1}{n^2}\right).$$

Using the formula

$$\sum_{l=1}^{\infty}\frac{sin(l\alpha)}{l} = \frac{\pi-\alpha}{2}, \qquad 0<\alpha<\pi,$$

we find

$$\gamma_1 = -\frac{1}{4\pi^2 n}\int_{\varepsilon}^{\pi-\varepsilon} f(\alpha)ctg\alpha(\pi-\alpha)d\alpha + O\left(\frac{1}{n^2}\right) + O\left(\frac{1}{n^2}\right).$$

$$\gamma_2 = O(1)\sum_{k=1,k\neq n}^{\infty} \frac{(2k+1)(2n+1)}{|k-n|(k+n+1)k^{1/2}n^{3/2}(k+n+2)} \leq O(1)\sum_{k=1,k\neq n}^{\infty}\frac{k^{1/2}}{n|k^2-n^2|} = O\left(\frac{\varepsilon\ln n}{n^2}\right)$$

.

As a result, we have

$$\alpha_n^5(p) = -\frac{1}{4\pi^2 n}\int_{\varepsilon}^{\pi-\varepsilon} f(\alpha)ctg\alpha(\pi-\alpha)d\alpha + O\left(\frac{\varepsilon\ln n}{n^2}\right)$$

.

Let us note that the remaining terms are estimated similarly.

Summing up, we get

$$\alpha_n(p) = O(\varepsilon^2\, ln\,n) + O\left(\frac{\varepsilon\, ln\,n}{n}\right) + O\left(\frac{\varepsilon}{n^{3/2}}\right) + O\left(\frac{\varepsilon}{n}\right) + O\left(\frac{\varepsilon\, ln\,n}{n^2}\right) + O\left(\frac{\varepsilon\, ln\,n}{n^2}\right) + O\left(\frac{\varepsilon}{n^2}\right) +$$

$$+ O\left(\frac{\varepsilon\ln n}{n^2}\right) - \frac{1}{4\pi^2 n}\int_{\varepsilon}^{\pi-\varepsilon} f(\alpha)ctg\alpha(\pi-\alpha)d\alpha + O\left(\frac{\varepsilon\ln n}{n^2}\right) + O\left(\frac{\ln n}{n^{3/2}}\right) +$$

$$+ O\left(\frac{\ln\varepsilon\ln n}{n^{3/2}}\right) + O\left(\frac{\ln n}{n^3\varepsilon}\right) + O\left(\frac{\ln\varepsilon}{n^{3/2}}\right) + O\left(\frac{1}{\varepsilon n^{3/2}}\right) + O\left(\frac{1}{n^3\varepsilon^3}\right).$$

Supposing $\varepsilon = \frac{1}{n^{3/4}}$, we find

$$\alpha_n(p) = \frac{O(1)}{n}\int_{\varepsilon}^{\pi-\varepsilon} f(\alpha)ctg\alpha(\pi-\alpha)d\alpha + O\left(\frac{ln\,n}{n^{3/2}}\right).$$

Thus, the third amendment of the perturbation theory is

$$\beta_n(p) = \frac{1}{2\pi i}\frac{1}{3}Sp\left[\int_{l_n} - \int_{l_{n-1}}\right]\left[P(T-\lambda E)^{-1}\right]^3 d\lambda =$$

$$= \frac{1}{6\pi i}\left[\int_{l_n} - \int_{l_{n-1}}\right]\sum_{\substack{k,l=1\\ k\neq n,\ l\neq n}}^{\infty} \frac{(2n+1)(2k+1)(2l+1)}{(\lambda_n-\lambda)(\lambda_k-\lambda)(\lambda_l-\lambda)}d\lambda\left(\frac{1}{4\pi}\right)^3 \iint_{\Phi}\iint_{\Phi}\iint_{\Phi} p(\theta,\varphi)p(\theta',\varphi')p(\theta'',\varphi'')\times$$

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3. Practical suggestions

Theorem1. If p is a potential satisfying the Lipchitz condition, then for the eigenvalues of the Neumann operator with the potential on the projective plane the equality is true

$$\sum_{i=0}^{2n}\mu_{n,i}-n(n+1)(2n+1)+$$
$$\frac{1}{16\pi^2}\int_{\varepsilon}^{\pi-\varepsilon}f(\alpha)tg\alpha d\alpha=O\left(\frac{5\ln n}{n^{5/3}}\right).$$

Theorem2. The first regularized trace of the Neumann operator with a complex potential on the projective plane is equal to

$$\sum_{k=0}^{\infty}\left\{\sum_{i=0}^{2n}\mu_{n,i}-n(n+1)(2n+1)\right\}=$$
$$-\frac{3}{16\pi^4}\iint_{\Phi}p^4(\theta,\varphi)\cos\theta d\varphi d\theta.$$

In the article, a mathematical model for calculating the eigenvalues of differential operators with a potential on a projective plane is constructed on the basis of spectral theory.

The interest in such problems is constantly increasing due to the wide scope of their application [3] - [10].

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