TEST Engineering & Management

November-December 2019
ISSN: 0193-4120 Page No. 764 - 782

New Generalized Direct Two-Step Hybrid Block Methods with All Possible Combinations of Three Off-Step Points for Solving Second Order Ordinary Differential Equations

Kamarun Hizam Mansor, Zurni Omar

School of Quantitative Sciences, Universiti Utara Malaysia, 06010 Sintok, Kedah, Malaysia

Article Info
Volume 81
Page Number: 764- 782
Publication Issue:
November-December 2019

Article History
Article Received: *3 January 2019*
Revised: *25 March 2019*
Accepted: *28 July 2019*
Publication: *25 November 2019*

Abstract
New generalized two-step hybrid block methods with three off-step points for direct solution of second order ordinary differential equations are proposed. The locations of three off-step points in two-step interval are obtained through permutation. The main continuous schemes are derived by interpolating approximate solutions in the form of power series at two points in a two-step interval while the second derivative of the approximate solutions are collocated at all points in the given interval. Basic properties of the method such as order, zero stability, consistency and convergence are also established. The numerical results show that the developed methods produce better accuracy than several existing methods when solving the initial value problems of second order ordinary differential equations.

1. Introduction

The following initial value problems (IVPs) of second order ordinary differential equations (ODEs) are considered in this paper

$$y'' = f(x, y, y'), y(a) = \eta_0, y'(a) = \eta_1 \text{ and } x \in [a, b]. \quad \textbf{(1)}$$

Equation (1) can be solved by reduction methods ([1, 2]) but this approach requires extra computation compared to direct methods ([3]-[13]). It was noticed that most of the previous direct hybrid block methods were derived for specific off-step points and can only approximate numerical solution of (1) at one step point at a time ([14] – [16]). Recently, Mansor et al.[17] developed a two-step hybrid block method with generalized one off-step point within each step to find the direct solution of second order ordinary differential equations. Although this method is capable of finding numerical solution and produces better accuracy if compared to the previous methods, it is only restricted to one off-step point for each step in the given interval.

November-December 2019
ISSN: 0193-4120 Page No. 764 - 782

This study, therefore, extends their work to three off-step points. In addition, this study also takes into account all possible combinations ([18, 20]) of three off-step points for solving second order ODEs which has not been considered before.

2. Development of the Method

This section discusses the derivation of new generalized two-step hybrid block methods with three off-step points for solving second order ordinary differential equations directly. All possible combinations of three-off step points are obtained through permutations as described in the following section.

2.1. Permutations Involving Three Off-Step Points

Consider a two-step interval containing the initial step point, x_n the interior step point, x_{n+1} and the last step point, x_{n+2} with off-step points $x_{n+p}, x_{n+q}, x_{n+r}$ where $0 < p < q < r < 2$.
The first step in deriving the proposed method is finding permutations all possible combination of four points which are located inside the interval $(x_n, x_{n+2}]$, i.e $x_{n+1}, x_{n+p}, x_{n+q}$ and x_{n+r}. In order to determine all possible combination of points, the following theorem is adopted.

Theorem 2.1 [20]. The number of permutations of p objects of which p_1 are of one kind, p_2 are of a second kind, …, p_k are of a kth kind, and $p_1 + p_2 + \cdots + p_k = p$ is

$$\frac{p!}{p_1! \cdot p_2! \cdot \ldots \cdot p_k!} \; . \qquad \textbf{(2)}$$

Based on the strategy previously described, the number of off-step points (p_1) considered in this study is 3, the number of interior step point (p_2) is 2 and the total number of points used inside the interval $(x_n, x_{n+2}]$ is $p = 4$. Applying (2) yields

$$\frac{4!}{1! \cdot 3!} = 4$$

which implies that three-off step points, $x_{n+p}, x_{n+q}, x_{n+r}$ with an interior step point,x_{n+1} can be arranged in four distinct combinations. All combinations can be obtained by using Mathematic a code as illustrated in the following figures.

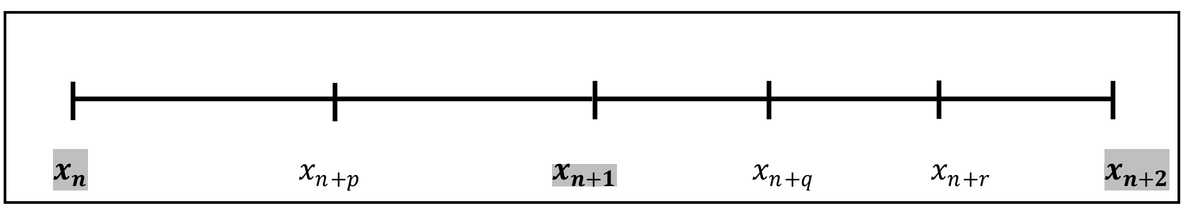

Figure 1: Combination 1: $0 < p < 1 < q < r < 2$.

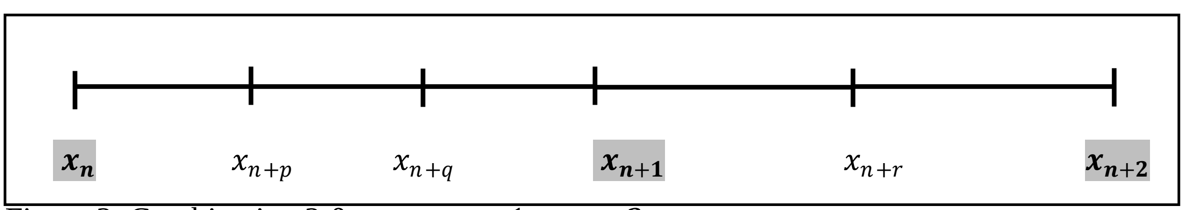

Figure 2: Combination 2:$0 < p < q < 1 < r < 2$.

November-December 2019
ISSN: 0193-4120 Page No. 764 - 782

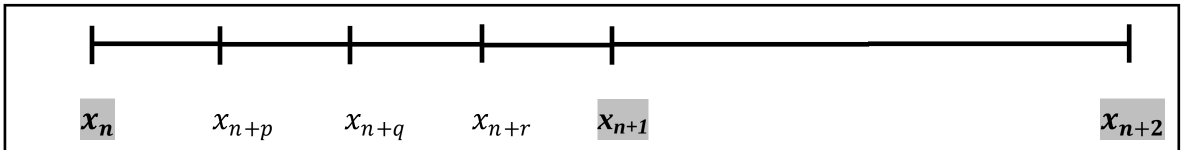

Figure 3: Combination 3:$0 < p < q < r < 1$.

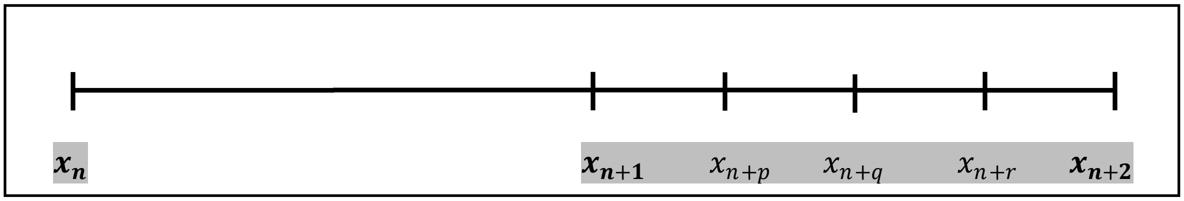

Figure 4: Combination 4:$1 < p < q < r < 2$.

2.2. Derivation of the Method

In this section, we will only demonstrate the derivation of two-step hybrid block method with three off-step points for Combination 1. The derivation for the other three combinations follows the same strategy.

Let the approximate solution of Equation (1) be the power series polynomial given by

$$y(x) = \sum_{j=0}^{i+c-1} a_j \left(\frac{x - x_n}{h}\right)^j \qquad \textbf{(3)}$$

where $x \in (x_n, x_{n+2}]$ for $n = 0,2,4,\dots,N-2$, i is the number of interpolation points which is equal to the order of differential equation, c is the number of collocation at all points in the interval and $h = x_{n+1} - x_n$ is a constant step size for the partition of interval $[a,b]$ defined by $a = x_0 < x_2 < x_4 < \cdots < x_n < x_{n+2} < \cdots < x_{N-2} = b$.

Now, differentiating Equation (3) twice yields

$$y''(x) = f(x, y, y') = \sum_{j=2}^{i+c-1} a_j \frac{j(j-1)}{h^2} \left(\frac{x - x_n}{h}\right)^{j-2}. \qquad \textbf{(4)}$$

Interpolating Equation(3) at x_{n+j}, $(j = 0, p)$ and collocating Equation(4) at all points, i.e x_{n+j}, $(j = 0, p, 1, q, r, 2)$in Combination 1 of two-step interval produces eight equations which can be written in the following matrix form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & p & p^2 & p^3 & p^4 & p^5 & p^6 & p^7 \\ 0 & 0 & \frac{2}{h^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{h^2} & \frac{6p}{h^2} & \frac{12p^2}{h^2} & \frac{20p^3}{h^2} & \frac{30p^4}{h^2} & \frac{42p^5}{h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{6}{h^2} & \frac{12}{h^2} & \frac{20}{h^2} & \frac{30}{h^2} & \frac{42}{h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{6q}{h^2} & \frac{12q^2}{h^2} & \frac{20q^3}{h^2} & \frac{30q^4}{h^2} & \frac{42q^5}{h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{6r}{h^2} & \frac{12r^2}{h^2} & \frac{20r^3}{h^2} & \frac{30r^4}{h^2} & \frac{42r^5}{h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{12}{h^2} & \frac{48}{h^2} & \frac{160}{h^2} & \frac{480}{h^2} & \frac{1344}{h^2} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+p} \\ f_n \\ f_{n+p} \\ f_{n+1} \\ f_{n+q} \\ f_{n+r} \\ f_{n+2} \end{pmatrix}. \quad \textbf{(5)}$$

Solving for a_j, $(j = 0,1,\ldots,7)$ in Equation (5) using Gaussian elimination and substitute these values in Equation (3) gives a continuous implicit hybrid two-step scheme with generalised three off-step points below

$$y(x) = \sum_{j=0,p} \alpha_j(x) y_{n+j} + \sum_{j=0}^{2} \beta_j(x) f_{n+j} + \sum_{j=p,q,r} \beta_j(x) f_{n+j}. \quad \textbf{(6)}$$

whose first derivative is

$$y'(x) = \sum_{j=0,p} \frac{\partial}{\partial x}\alpha_j(x) y_{n+j} + \sum_{j=0}^{2} \frac{\partial}{\partial x}\beta_j(x) f_{n+j} + \sum_{j=p,q,r} \frac{\partial}{\partial x}\beta_j(x) f_{n+j}. \quad \textbf{(7)}$$

Equation (6) is evaluated at the non-interpolating points, i.e at x_{n+j}, $(j = 1, q, r, 2)$ while Equation (7) is evaluated at all points, i.e at x_{n+j}, $(j = 0, p, 1, q, r, 2)$ to produce a discrete implicit hybrid two-step scheme with generalised three off-step points and its derivative at x_n. The discrete scheme and its derivative at x_n are then combined simultaneously to give a main block of new method in a matrix form as follows

$$A_{(1)}^{2[3]_2} Y_{m_{(1)}}^{2[3]_2} = B_{1_{(1)}}^{2[3]_2} R_{1_{(1)}}^{2[3]_2} + B_{2_{(1)}}^{2[3]_2} R_{2_{(1)}}^{2[3]_2} + h^2 \left[D_{(1)}^{2[3]_2} R_{3_{(1)}}^{2[3]_2} + E_{(1)}^{2[3]_2} R_{4_{(1)}}^{2[3]_2} \right] \quad \textbf{(8)}$$

where

$$Y_{m_{(1)}}^{2[3]_2} = \begin{pmatrix} y_{n+p} \\ y_{n+1} \\ y_{n+q} \\ y_{n+r} \\ y_{n+2} \end{pmatrix}, R_{1_{(1)}}^{2[3]_2} = \begin{pmatrix} y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}, R_{2_{(1)}}^{2[3]_2} = \begin{pmatrix} y'_{n-4} \\ y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_n \end{pmatrix}, R_{3_{(1)}}^{2[3]_2} = \begin{pmatrix} f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}, R_{4_{(1)}}^{2[3]_2} = \begin{pmatrix} f_{n+p} \\ f_{n+1} \\ f_{n+q} \\ f_{n+r} \\ f_{n+2} \end{pmatrix},$$

$$A_{(1)}^{2[3]_2} = \begin{pmatrix} \frac{-1}{p} & 1 & 0 & 0 & 0 \\ \frac{-q}{p} & 0 & 1 & 0 & 0 \\ \frac{-r}{p} & 0 & 0 & 1 & 0 \\ \frac{-2}{p} & 0 & 0 & 0 & 1 \\ \frac{-1}{hp} & 0 & 0 & 0 & 0 \end{pmatrix}, B_{1_{(1)}}^{2[3]_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{(p-1)}{p} \\ 0 & 0 & 0 & 0 & \frac{(p-q)}{p} \\ 0 & 0 & 0 & 0 & \frac{(p-r)}{p} \\ 0 & 0 & 0 & 0 & \frac{(p-2)}{p} \\ 0 & 0 & 0 & 0 & \frac{-1}{hp} \end{pmatrix},$$

$$B_{2_{(1)}}^{2[3]_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, D_{(1)}^{2[3]_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & D_{15} \\ 0 & 0 & 0 & 0 & D_{25} \\ 0 & 0 & 0 & 0 & D_{35} \\ 0 & 0 & 0 & 0 & D_{45} \\ 0 & 0 & 0 & 0 & D_{55} \end{pmatrix},$$

$$E_{(1)}^{2[3]_2} = \begin{pmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} \\ E_{21} & E_{22} & E_{23} & E_{24} & E_{25} \\ E_{31} & E_{32} & E_{33} & E_{34} & E_{35} \\ E_{41} & E_{42} & E_{43} & E_{44} & E_{45} \\ E_{51} & E_{52} & E_{53} & E_{54} & E_{55} \end{pmatrix}.$$

Note: The elements of $D_{(1)}^{2[3]_2}$ and $E_{(1)}^{2[3]_2}$ are given in Appendix A.

Multiplying Equation (8) by the inverse of $A^{2[3]_2}$ gives a new generalized two-step hybrid block with three off-step points based on Combination 1 as below

$$I_{(1)}^{2[3]_2} Y_{m_{(1)}}^{2[3]_2} = \bar{B}_{1_{(1)}}^{2[3]_2} R_{1_{(1)}}^{2[3]_2} + \bar{B}_{2_{(1)}}^{2[3]_2} R_{2_{(1)}}^{2[3]_2} + h^2 \left[\bar{D}_{(1)}^{2[3]_2} R_{3_{(1)}}^{2[3]_2} + \bar{E}_{(1)}^{2[3]_2} R_{4_{(1)}}^{2[3]_2} \right] \quad \textbf{(9)}$$

where

$$I_{(1)}^{2[3]_2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \bar{B}_{1_{(1)}}^{2[3]_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \bar{B}_{2_{(1)}}^{2[3]_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & hp \\ 0 & 0 & 0 & 0 & h \\ 0 & 0 & 0 & 0 & hq \\ 0 & 0 & 0 & 0 & hr \\ 0 & 0 & 0 & 0 & 2h \end{pmatrix},$$

November-December 2019
ISSN: 0193-4120 Page No. 764 - 782

$$\bar{D}_{(1)}^{2[3]_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & \bar{D}_p \\ 0 & 0 & 0 & 0 & \bar{D}_1 \\ 0 & 0 & 0 & 0 & \bar{D}_q \\ 0 & 0 & 0 & 0 & \bar{D}_r \\ 0 & 0 & 0 & 0 & \bar{D}_2 \end{pmatrix}, \bar{E}_{(1)}^{2[3]_2} = \begin{pmatrix} \bar{E}_{p1} & \bar{E}_{p2} & \bar{E}_{p3} & \bar{E}_{p4} & \bar{E}_{p5} \\ \bar{E}_1 & \bar{E}_2 & \bar{E}_3 & \bar{E}_4 & \bar{E}_5 \\ \bar{E}_{q1} & \bar{E}_{q2} & \bar{E}_{q3} & \bar{E}_{q4} & \bar{E}_{q5} \\ \bar{E}_{r1} & \bar{E}_{r2} & \bar{E}_{r3} & \bar{E}_{r4} & \bar{E}_5 \\ \bar{E}_2 & \bar{E}_4 & \bar{E}_6 & \bar{E}_8 & \bar{E}_{10} \end{pmatrix}.$$

which then leads to

$$\begin{pmatrix} y_{n+p} \\ y_{n+1} \\ y_{n+q} \\ y_{n+r} \\ y_{n+2} \end{pmatrix} = \begin{pmatrix} 1 & hp \\ 1 & h \\ 1 & hq \\ 1 & hr \\ 1 & 2h \end{pmatrix} \begin{pmatrix} y_n \\ y'_n \end{pmatrix} + h^2 \left[\begin{pmatrix} \bar{D}_p \\ \bar{D}_1 \\ \bar{D}_q \\ \bar{D}_r \\ \bar{D}_2 \end{pmatrix} (f_n) + \begin{pmatrix} \bar{E}_{p1} & \bar{E}_{p2} & \bar{E}_{p3} & \bar{E}_{p4} & \bar{E}_{p5} \\ \bar{E}_1 & \bar{E}_2 & \bar{E}_3 & \bar{E}_4 & \bar{E}_5 \\ \bar{E}_{q1} & \bar{E}_{q2} & \bar{E}_{q3} & \bar{E}_{q4} & \bar{E}_{q5} \\ \bar{E}_{r1} & \bar{E}_{r2} & \bar{E}_{r3} & \bar{E}_{r4} & \bar{E}_5 \\ \bar{E}_2 & \bar{E}_4 & \bar{E}_6 & \bar{E}_8 & \bar{E}_{10} \end{pmatrix} \begin{pmatrix} f_{n+p} \\ f_{n+1} \\ f_{n+q} \\ f_{n+r} \\ f_{n+2} \end{pmatrix} \right],$$

with

$$\bar{B}_{1_{(1)}}^{2[3]_2} R_{1_{(1)}}^{2[3]_2} + \bar{B}_{2_{(1)}}^{2[3]_2} R_{2_{(1)}}^{2[3]_2} = \begin{pmatrix} 1 & hp \\ 1 & h \\ 1 & hq \\ 1 & hr \\ 1 & 2h \end{pmatrix} \begin{pmatrix} y_n \\ y'_n \end{pmatrix} \text{ and } \bar{D}_{(1)}^{2[3]_2} R_{3_{(1)}}^{2[3]_2} = \begin{pmatrix} \bar{D}_p \\ \bar{D}_1 \\ \bar{D}_q \\ \bar{D}_r \\ \bar{D}_2 \end{pmatrix} (f_n).$$

Expressing (9) in simultaneous equations gives

$$y_{n+p} = y_n + hpy'_n + h^2[\bar{D}_p f_n + \bar{E}_{p1} f_{n+p} + \bar{E}_{p2} f_{n+1} + \bar{E}_{p3} f_{n+q} + \bar{E}_{p4} f_{n+r} + \bar{E}_{p5} f_{n+2}] \quad \textbf{(10)}$$
$$y_{n+1} = y_n + hy'_n + h^2[\bar{D}_1 f_n + \bar{E}_1 f_{n+p} + \bar{E}_2 f_{n+1} + \bar{E}_3 f_{n+q} + \bar{E}_4 f_{n+r} + \bar{E}_5 f_{n+2}] \quad \textbf{(11)}$$
$$y_{n+q} = y_n + hqy'_n + h^2[\bar{D}_q f_n + \bar{E}_{q1} f_{n+p} + \bar{E}_{q2} f_{n+1} + \bar{E}_{q3} f_{n+q} + \bar{E}_{q4} f_{n+r} + \bar{E}_{q5} f_{n+2}] \quad \textbf{(12)}$$
$$y_{n+r} = y_n + hry'_n + h^2[\bar{D}_r f_n + \bar{E}_{r1} f_{n+p} + \bar{E}_{r2} f_{n+1} + \bar{E}_{r3} f_{n+q} + \bar{E}_{r4} f_{n+r} + \bar{E}_{r5} f_{n+2}] \quad \textbf{(13)}$$
$$y_{n+2} = y_n + 2hy'_n + h^2[\bar{D}_2 f_n + \bar{E}_2 f_{n+p} + \bar{E}_4 f_{n+1} + \bar{E}_6 f_{n+q} + \bar{E}_8 f_{n+r} + \bar{E}_{10} f_{n+2}] \quad \textbf{(14)}$$

Note: The elements of $\bar{D}_{(1)}^{2[3]_2}$ and $\bar{E}_{(1)}^{2[3]_2}$ are shown in Appendix B.

Combining (8) with the first derivative of a discrete scheme at x_{n+j}, $(j = p,1,q,r,2$ yields the first derivative of the block in matrix form as below

$$\begin{pmatrix} y'_{n+p} \\ y'_{n+1} \\ y'_{n+q} \\ y'_{n+r} \\ y'_{n+2} \end{pmatrix} = \begin{pmatrix} y'_n \\ y'_n \\ y'_n \\ y'_n \\ y'_n \end{pmatrix} + h \left[\begin{pmatrix} \acute{D}_p \\ \acute{D}_1 \\ \acute{D}_q \\ \acute{D}_r \\ \acute{D}_2 \end{pmatrix} (f_n) + \begin{pmatrix} \acute{E}_{p1} & \acute{E}_{p2} & \acute{E}_{p3} & \acute{E}_{p4} & \acute{E}_{p5} \\ \acute{E}_1 & \acute{E}_2 & \acute{E}_3 & \acute{E}_4 & \acute{E}_5 \\ \acute{E}_{q1} & \acute{E}_{q2} & \acute{E}_{q3} & \acute{E}_{q4} & \acute{E}_{q5} \\ \acute{E}_{r1} & \acute{E}_{r2} & \acute{E}_{r3} & \acute{E}_{r4} & \acute{E}_{r5} \\ \acute{E}_2 & \acute{E}_4 & \acute{E}_6 & \acute{E}_8 & \acute{E}_{10} \end{pmatrix} \begin{pmatrix} f_{n+p} \\ f_{n+1} \\ f_{n+q} \\ f_{n+r} \\ f_{n+2} \end{pmatrix} \right]. \quad \textbf{(15)}$$

November-December 2019
ISSN: 0193-4120 Page No. 764 - 782

Similarly, Equation (15) can be written as simultaneous equations as follows

$$y'_{n+p} = y'_n + h[\acute{D}_p f_n + \acute{E}_{p1} f_{n+p} + \acute{E}_{p2} f_{n+1} + \acute{E}_{p3} f_{n+q} + \acute{E}_{p4} f_{n+r} + \acute{E}_{p5} f_{n+2}] \quad \textbf{(16)}$$
$$y'_{n+1} = y'_n + h[\acute{D}_1 f_n + \acute{E}_1 f_{n+p} + \acute{E}_2 f_{n+1} + \acute{E}_3 f_{n+q} + \acute{E}_4 f_{n+r} + \acute{E}_5 f_{n+2}] \quad \textbf{(17)}$$
$$y'_{n+q} = y'_n + h[\acute{D}_q f_n + \acute{E}_{q1} f_{n+p} + \acute{E}_{q2} f_{n+1} + \acute{E}_{q3} f_{n+q} + \acute{E}_{q4} f_{n+r} + \acute{E}_{q5} f_{n+2}] \quad \textbf{(18)}$$
$$y'_{n+r} = y'_n + h[\acute{D}_r f_n + \acute{E}_{r1} f_{n+p} + \acute{E}_{r2} f_{n+1} + \acute{E}_{r3} f_{n+q} + \acute{E}_{r4} f_{n+r} + \acute{E}_{r5} f_{n+2}] \quad \textbf{(19)}$$
$$y'_{n+2} = y'_n + h[\acute{D}_2 f_n + \acute{E}_2 f_{n+p} + \acute{E}_4 f_{n+1} + \acute{E}_6 f_{n+q} + \acute{E}_8 f_{n+r} + \acute{E}_{10} f_{n+2}] \quad \textbf{(20)}$$

Note: The elements of $\acute{D}_p, \dots, \acute{D}_2$ and $\acute{E}_{p1}, \dots, \acute{E}_{10}$ are shown in Appendix C.
The other three combinations will produce similar equations as given in (10) – (14) and (16) – (20) except the position of subscripts change according to the position of the collocation points in the related combinations. For example, a new generalized two-step hybrid block with three off-step points based on Combination 2 can be obtained by interchanging the position of (11) with (12), and (17) with (18). As a result, we get the main block

$$y_{n+p} = y_n + hpy'_n + h^2[\bar{D}_p f_n + \bar{E}_{p1} f_{n+p} + \bar{E}_{p2} f_{n+1} + \bar{E}_{p3} f_{n+q} + \bar{E}_{p4} f_{n+r} + \bar{E}_{p5} f_{n+2}] \quad \textbf{(21)}$$
$$y_{n+q} = y_n + hqy'_n + h^2[\bar{D}_q f_n + \bar{E}_{q1} f_{n+p} + \bar{E}_{q2} f_{n+1} + \bar{E}_{q3} f_{n+q} + \bar{E}_{q4} f_{n+r} + \bar{E}_{q5} f_{n+2}] \quad \textbf{(22)}$$
$$y_{n+1} = y_n + hy'_n + h^2[\bar{D}_1 f_n + \bar{E}_1 f_{n+p} + \bar{E}_2 f_{n+1} + \bar{E}_3 f_{n+q} + \bar{E}_4 f_{n+r} + \bar{E}_5 f_{n+2}] \quad \textbf{(23)}$$
$$y_{n+r} = y_n + hry'_n + h^2[\bar{D}_r f_n + \bar{E}_{r1} f_{n+p} + \bar{E}_{r2} f_{n+1} + \bar{E}_{r3} f_{n+q} + \bar{E}_{r4} f_{n+r} + \bar{E}_{r5} f_{n+2}] \quad \textbf{(24)}$$
$$y_{n+2} = y_n + 2hy'_n + h^2[\bar{D}_2 f_n + \bar{E}_2 f_{n+p} + \bar{E}_4 f_{n+1} + \bar{E}_6 f_{n+q} + \bar{E}_8 f_{n+r} + \bar{E}_{10} f_{n+2}] \quad \textbf{(25)}$$

with its first derivatives

$$y'_{n+p} = y'_n + h[\acute{D}_p f_n + \acute{E}_{p1} f_{n+p} + \acute{E}_{p2} f_{n+1} + \acute{E}_{p3} f_{n+q} + \acute{E}_{p4} f_{n+r} + \acute{E}_{p5} f_{n+2}] \quad \textbf{(26)}$$
$$y'_{n+q} = y'_n + h[\acute{D}_q f_n + \acute{E}_{q1} f_{n+p} + \acute{E}_{q2} f_{n+1} + \acute{E}_{q3} f_{n+q} + \acute{E}_{q4} f_{n+r} + \acute{E}_{q5} f_{n+2}] \quad \textbf{(27)}$$
$$y'_{n+1} = y'_n + h[\acute{D}_1 f_n + \acute{E}_1 f_{n+p} + \acute{E}_2 f_{n+1} + \acute{E}_3 f_{n+q} + \acute{E}_4 f_{n+r} + \acute{E}_5 f_{n+2}] \quad \textbf{(28)}$$
$$y'_{n+r} = y'_n + h[\acute{D}_r f_n + \acute{E}_{r1} f_{n+p} + \acute{E}_{r2} f_{n+1} + \acute{E}_{r3} f_{n+q} + \acute{E}_{r4} f_{n+r} + \acute{E}_{r5} f_{n+2}] \quad \textbf{(29)}$$
$$y'_{n+2} = y'_n + h[\acute{D}_2 f_n + \acute{E}_2 f_{n+p} + \acute{E}_4 f_{n+1} + \acute{E}_6 f_{n+q} + \acute{E}_8 f_{n+r} + \acute{E}_{10} f_{n+2}] \quad \textbf{(30)}$$

2.3 Order of the Methods

The linear difference operator L associated with (9) is defined as

$$L[y(x);h] = I_{(1)}^{2[3]_2} Y_{m_{(1)}}^{2[3]_2} - \bar{B}_{1_{(1)}}^{2[3]_2} R_{1_{(1)}}^{2[3]_2} - \bar{B}_{2_{(1)}}^{2[3]_2} R_{2_{(1)}}^{2[3]_2} - h^2 \left[\bar{D}_{(1)}^{2[3]_2} R_{3_{(1)}}^{2[3]_2} + \bar{E}_{(1)}^{2[3]_2} R_{4_{(1)}}^{2[3]_2}\right] \quad \textbf{(31)}$$

where $y(x)$ is an arbitrary test function continuously differentiable on $[a, b]$. Now, expanding $Y_{m_{(1)}}^{2[3]_2}$ and $R_{4_{(1)}}^{2[3]_2}$ components in Taylors series and collecting their terms in powers of h gives

November-December 2019
ISSN: 0193-4120 Page No. 764 - 782

$$L[y(x);h] = \bar{C}_{0_{(1)}}^{2[3]_2} y(x) + \bar{C}_{1_{(1)}}^{2[3]_2} h y'(x) + \bar{C}_{2_{(1)}}^{2[3]_2} h^2 y''(x) + \cdots \quad \textbf{(32)}$$

Definition 2.1 Hybrid block method (9) and associated linear operator in (31) are said to be of order $\boldsymbol{d} = [d_1, d_2, d_3, d_4, d_5]^T$ if $\bar{C}_{0_{(1)}}^{2[3]_2} = \bar{C}_{1_{(1)}}^{2[3]_2} = \bar{C}_{2_{(1)}}^{2[3]_2} = \cdots = \bar{C}_{d+1_{(1)}}^{2[3]_2} = 0$ and $\bar{C}_{d+2_{(1)}}^{2[3]_2} \neq 0$ with error vector constants $\bar{C}_{d+2_{(1)}}^{2[3]_2}$.

Expanding (31) in Taylor series about x_n gives the order of method which is $[6,6,6,6,6]^T$. So, the new method is consistent since its order is greater than 1.

2.4 Zero Stability

The hybrid block method in (9) and it derivative (15) is said to be *zero-stable* if no root of the first characteristic polynomial $\rho(\tau) = \left|\tau I_{5\times5} - \bar{B}_{1_{(1)}}^{2[3]_2}\right|$ is having a modulus greater than one and every root of modulus one is simple, where $I_{5\times5}$ is identity matrix and $\bar{B}_{1_{(1)}}^{2[3]_2}$ is the coefficients matrix of y_n function. Equating $\rho(\tau) = 0$, we have

$$= \left| \tau \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| = \tau^4(\tau - 1) = 0$$

which implies $\tau = 0,0,0,0,1$. Hence, our method is zero stable.

2.5 Consistency and Convergence

Theorem 2.2[1]: Consistency and zero stability are sufficient conditions for a linear multistep method to be convergent.

Since the new method in (9) is consistent and zero stable, this implies that it is also convergent.

Remark: Other new methods based on Combination 2, 3 and 4 share the same properties as the method derived using Combination 1.

3. Implementation

Taylor Series are employed once to each developed methods to produce predicted initial values for $y_{n+i}, i = 1,2,p,q,r$ where $0 < p < q < r < 2$. The computation is done in a block. In the first block, the obtained values are substituted in the developed methods to get the corrected values of $y_{n+i}, i = 1,2,p,q,r$. For the second block, the value at x_{n+2} is used as the intial value and substituted in the proposed methods to yield the approximate solution. This process is repeated until we reach the last point in the given interval.

4. Numerical Experiments

The following second order ODEs problems are tested for $0 < x < 10$ as capability of our methods in finding numerical solution. However, we use the same value of x as employed in other existing methods for the purpose of comparison.

November-December 2019
ISSN: 0193-4120 Page No. 764 - 782

The following notations are used in the tables:

G2S3PBHC1: Generalized two-step hybrid block method with three off-step points based on Combination 1.
G2S3PBHC2: Generalized two-step hybrid block method with three off-step points based on Combination 2.
G2S3PBHC3: Generalized two-step hybrid block method with three off-step points based on Combination 3.
G2S3PBHC4: Generalized two-step hybrid block method with three off-step points based on Combination 4.
AE: Absolute error.
OSPC1P1: Off-step points used in G2S3PBHC1, $p = \frac{1}{16}, q = \frac{5}{4}, r = \frac{4}{3}$ for Problem 1
OSPC2P1: Off-step points used in G2S3PBHC2, $p = \frac{1}{16}, q = \frac{1}{3}, r = \frac{4}{3}$ for Problem 1
OSPC3P1: Off-step points used in G2S3PBHC3, $p = \frac{1}{16}, q = \frac{1}{3}, r = \frac{1}{2}$ for Problem 1
OSPC4P1: Off-step points used in G2S3PBHC4, $p = \frac{17}{16}, q = \frac{5}{4}, r = \frac{4}{3}$ for Problem 1
OSPC1P2: Off-step points used in G2S3PBHC1, $p = \frac{1}{16}, q = \frac{5}{4}, r = \frac{4}{3}$ for Problem 2
OSPC2P2: Off-step points used in G2S3PBHC2, $p = \frac{8}{10}, q = \frac{95}{100}, r = \frac{1003}{1000}$ for Problem 2
OSPC3P2: Off-step points used in G2S3PBHC3, $p = \frac{9}{10}, q = \frac{94}{100}, r = \frac{95}{100}$ for Problem 2
OSPC4P2: Off-step points used in G2S3PBHC4, $p = \frac{1002}{1000}, q = \frac{5}{4}, r = \frac{3}{2}$ for Problem 2
OSPC1P3: Off-step points used in G2S3PBHC1, $p = \frac{1}{16}, q = \frac{5}{4}, r = \frac{4}{3}$ for Problem 3
OSPC2P3: Off-step points used in G2S3PBHC2, $p = \frac{1}{4}, q = \frac{1}{2}, r = \frac{19}{10}$ for Problem 3
OSPC3P3: Off-step points used in G2S3PBHC3, $p = \frac{1}{16}, q = \frac{1}{3}, r = \frac{1}{2}$ for Problem 3
OSPC4P3: Off-step points used in G2S3PBHC4, $p = \frac{17}{16}, q = \frac{5}{4}, r = \frac{4}{3}$ for Problem 3
OSPC1P4: Off-step points used in G2S3PBHC1, $p = \frac{1}{16}, q = \frac{5}{4}, r = \frac{4}{3}$ for Problem 4
OSPC2P4: Off-step points used in G2S3PBHC2, $p = \frac{1}{4}, q = \frac{1}{3}, r = \frac{4}{3}$ for Problem 4
OSPC3P4: Off-step points used in G2S3PBHC3, $p = \frac{1}{16}, q = \frac{1}{3}, r = \frac{1}{2}$ for Problem 4
OSPC4P4: Off-step points used in G2S3PBHC4, $p = \frac{4}{3}, q = \frac{5}{3}, r = \frac{19}{10}$ for Problem 4
d: Order of the method.

4.1 Tested Problems

Problem 1: $y'' - y = 0,\ y(0) = 1, y'(0) = 1, 0 < x < 1$ with $h = \frac{1}{10}$.
Exact solution: $y(x) = e^x$.
Source: [21].

Table 1: Comparison of errors obtained by new method with Sagir (2012) for Problem 1

x	New method				AE in Sagir (2012) $d = 5$
	AE in G2S3PBHC1	AE in G2S3PBHC2	AE in G2S3PBHC3	AE in G2S3PBHC4	
	OSPC1P1	OSPC2P1	OSPC3P1	OSPC4P1	
0.1	4.187761e-13	6.528111e-14	1.953993e-14	1.370237e-12	-
0.2	1.073142e-12	7.382983e-13	2.569944e-12	3.414158e-12	-
0.3	1.864287e-12	2.777778e-12	9.844792e-12	7.201795e-12	5.7600 e-10
0.4	2.956524e-12	5.586642e-12	2.040546e-11	1.186873e-11	1.6413 e-09
0.5	4.232170e-12	1.011125e-11	3.686140e-11	1.874234e-11	1.7001 e-09
0.6	5.895506e-12	1.564282e-11	5.757905e-11	2.678857e-11	2.3905 e-09
0.7	7.805756e-12	2.335865e-11	8.582424e-11	3.765344e-11	3.4705 e-09
0.8	1.021538e-11	3.241363e-11	1.196825e-10	5.009859e-11	4.4925 e-09
0.9	1.295675e-11	4.426992e-11	1.632294e-10	6.616085e-11	4.1569 e-09
1.0	1.634293e-11	5.792034e-11	2.142171e-10	8.435475e-11	4.4590 e-09

Problem 2: $y'' + \left(\frac{6}{x}\right)y' + \left(\frac{4}{x^2}\right)y = 0,\ y(1) = 1, y'(1) = 1$ with $h = \frac{1}{320}$.
Exact solution: $y(x) = \frac{5}{3x} - \frac{4}{x^2}$.
Source: [22]

Table 2: Comparison of errors obtained by new method withAnake (2011) for Problem 2

x	New method				AE in Anake (2011) $d = 5$
	AE in G2S3PBHC1	AE in G2S3PBHC2	AE in G2S3PBHC3	AE in G2S3PBHC4	
	OSPC1P2	OSPC2P2	OSPC3P2	OSPC4P2	
1.0031	4.662937e-15	1.998401e-13	2.731149e-14	9.547918e-15	7.7009510e-10
1.0063	1.199041e-14	2.697842e-13	4.662937e-14	2.442491e-14	7.1779915e-08
1.0094	1.754152e-14	4.702905e-13	7.438494e-14	4.662937e-14	1.9176578e-07
1.0125	2.531308e-14	5.464518e-13	9.414691e-14	7.371881e-14	3.5708470e-07
1.0156	3.130829e-14	7.456258e-13	1.219025e-13	1.068035e-13	5.6571030e-07
1.0188	3.974598e-14	8.271162e-13	1.421085e-13	1.447731e-13	8.1569016e-07
1.0219	4.596323e-14	1.026734e-12	1.676437e-13	1.887379e-13	1.1051428e-06
1.0250	5.506706e-14	1.112888e-12	1.860734e-13	2.362555e-13	1.4322554e-06
1.0281	6.239453e-14	1.311173e-12	2.120526e-13	2.888800e-13	1.7952815e-06
1.0313	7.172041e-14	1.401768e-12	2.311484e-13	3.450573e-13	2.1925381e-06

Problem 3: $y'' - x(y')^2 = 0,\ y(0) = 1, y'(0) = \frac{1}{2}, 0 < x < 1$ with $h = \frac{1}{10}$.
Exact solution: $y(x) = 1 + \frac{1}{2} ln\left|\frac{2+x}{2-x}\right|$.

TEST Engineering & Management

November-December 2019
ISSN: 0193-4120 Page No. 764 - 782

Source: [23]

Table 3: Comparison of errors obtained by new method Kuboye (2015) for Problem 3

x	New method				Kuboye (2015) d =7
	AE in G2S3PBHC1	AE in G2S3PBHC2	AE in G2S3PBHC3	AE in G2S3PBHC4	
	OSPC1P3	OSPC2P3	OSPC3P3	OSPC4P3	
0.1	2.718270e-12	4.107825e-13	9.636736e-14	1.007150e-11	9.577668e-10
0.2	6.826539e-12	4.367839e-12	1.508949e-11	2.531642e-11	2.368709e-09
0.3	2.077449e-11	1.471157e-11	5.678347e-11	8.140799e-11	3.732243e-09
0.4	4.170730e-11	4.617018e-11	1.707496e-10	1.595715e-10	5.475119e-09
0.5	1.352718e-10	1.467546e-10	3.611877e-10	3.100911e-10	1.142189e-08
0.6	2.540645e-10	3.146323e-10	7.713679e-10	5.286789e-10	4.567944e-08
0.7	7.043075e-10	7.956353e-10	1.305181e-09	8.810028e-10	2.055838e-06
0.8	1.258443e-09	1.517135e-09	2.541271e-09	1.452342e-09	4.248299e-06
0.9	3.301310e-09	3.596813e-09	3.813871e-09	2.336876e-09	6.660458e-06
1.0	5.853812e-09	6.711578e-09	7.692168e-09	4.038478e-09	9.445166e-06

Problem 4: $y'' - y' = 0,\ y(0) = 1, y'(0) = -1, 0 < x < 1$ with $h = \frac{1}{10}$.
Exact solution: $y(x) = 1 - e^x$.
Source: [24]

Table 4: Comparison of errors obtained by new method Mohammad (2010) for Problem 4

x	New method				Mohammaed (2010) d =7
	AE in G2S3PBHC1	AE in G2S3PBHC2	AE in G2S3PBHC3	AE in G2S3PBHC4	
	OSPC1P4	OSPC2P4	OSPC3P4	OSPC4P4	
0.1	4.309053e-13	6.596113e-14	1.984524e-14	4.382217e-12	5.7260e-06
0.2	1.186523e-12	7.233936e-13	2.552181e-12	1.151526e-11	6.6391e-06
0.3	8.678003e-12	4.003908e-12	2.927214e-12	1.607892e-11	7.0283e-06
0.4	1.729872e-11	8.514522e-12	6.509904e-12	2.391931e-11	7.4539e-06
0.5	3.495615e-11	2.016076e-11	7.455703e-12	2.852985e-11	7.8935e-06
0.6	5.488765e-11	3.216027e-11	1.237055e-11	3.704104e-11	8.1942e-06
0.7	8.684542e-11	5.355760e-11	1.412137e-11	4.149525e-11	8.1810e-06
0.8	1.226736e-10	7.614065e-11	2.078204e-11	5.059020e-11	8.1810e-06
0.9	1.743983e-10	1.110356e-10	2.364797e-11	5.459322e-11	8.1730e-06
1.0	2.321852e-10	1.483007e-10	3.258749e-11	6.411316e-11	8.1650e-06

5. Discussion and Conclusion

New two-step hybrid block methods with generalized three off-step points based on four combinations of two-step interval to directly solve the second order ordinary differential equations have been successfully developed. Besides being capable of approximating numerical at two step point simultaneously, the proposed methods are more flexible since they consider generalized off-step points in the derivation. In terms of accuracy, the new methods have performed better than the existing hybrid block methods when solving the same problems.

6. Acknowledgments

The financial support provided by Ministry of Higher, Malaysia under FRGS grant (SO Code: 13818) is highly appreciated.

References

[1] Henrici,P. Discrete Variable Methods in Ordinary Deferential Equations ;John Wiley & Sons, New York, NY, USA, 1962.
[2] Lambert, J. D. Computational methods in ordinary deferential equations, (1973).
[3] Omar, Z., Sulaiman, M., Saman, M.Y., Evans, D.J.Parallel R-point explicit block method for solving second order ordinary differential equations directly. International Journal of Computer Mathematics. 2002.79(3), 289-298
[4] James, A., Adesanya, A., & Joshua, S. Continuous block method for the solution of second order initial value problems of ordinary differential equation. International Journal of Pure and Applied Mathematics. 2013, 83(3), 405–416.
[5] Abdelrahim, R.; Omar, Z. Solving third order ordinary deferential equations using hybrid block method of order five. International Journal of Applied Engineering Research. 2015, 10, 44307 -44310.
[6] Omar, Z; Abdelrahim, R. Application of single step with three generalized hybrid points block method for solving third order ordinary deferential equations. International Journal of Applied Engineering Research. 2016, 9, 2705 - 2717.
[7] Awoyemi, D. Algorithmic collocation approach for direct solution of fourth-order initial-value problems of ordinary diffrential equations. Int. J. Comput. Math. 2005, 82, 321329.
[8] Yap, L; Ismail. F. Block hybrid collocation method with application to fourth order differential equations, mathematical problems in engineering. 2015, 1- 6.
[9] Hussain, K.; Ismail, F.; Senu1, N. Runge-kutta type methods for directly solving special fourth-order ordinary deferential equations. Mathematical Problems in Engineering. 2015.
[10] Jator,S.N. Numerical integrators for fourth order initial and boundary value problems. International Journal of Pure and Applied Mathematics. 2008, 47, 563 -576.
[11] Awoyemi, D. O. Algorithmic collocation approach for direct solution of fourth-order Initial-value problems of ordinary deferential equations. International Journal of Computer Mathematics. 2005, 47, 321-329.
[12] Omar, Z., Suleiman, M.B.Parallel two-point explicit block method for solving high-order ordinary differential equations.International Journal of Simulation and Process Modelling.2006. 2(3-4), 227-231
[13] Waeleh, N.; Majid, Z.; Ismail, F.; Suleiman, M. Numerical solution of higher order ordinary differential equations by direct block code. J. Math. Stat. 2012, 8, 7781.
[14] Adesanya, A., Ibrahim, Y., Abdulkadi, B., & Anake, T. Order four continuous hybrid block method for the solutions of second order ordinary differential equations. Journal of Mathematical and Computational Science. 2014 4(5), 817–825.

[15] Anake, T. A., Awoyemi, D. O., & Adesanya, A. One-step implicit hybrid block method for the direct solution of general second order ordinary differential equations. IAENG International Journal of Applied Mathematics, 42(4), 224–228 (2012a).

[16] Anake, T. A., Awoyemi, D. O., & Adesanya, A. A one step method for the solution of general second order ordinary differential equations. International Journal of science and Technology, 2(4), 159–163 (2012b).

[17] Mansor, K. H, Omar, Z, Rohni, M, A. Generalized two-step hybrid block method with one off-step point for solving second order ordinary differential equations directly. Journal of Advanced Research in Dynamical and Control Systems. 2018, 13, 1 – 17.

[18] Lockwood, E., Swinyard, C., & Caughman, J. S. (2013). Reinventing Permutation and Combination. Journal of Mathematical Behavior, *32*, 251-265.

[19] Falih A. M. Aldosray. Permutations and combinations. JECET. 2019; Sec. C; Vol.8. No.2, 104-107.

[20] Miller. I, Miller. M. John E. Freund's mathematical statistics with applications. Pearson New International Edition. 2014, 8thEdition.

[21] Sagir, A. An accurate computation of block hybrid method for solving stiff ordinary differential equations. Journal of Mathematics. 2012, 4, 18–21.

[22] Anake, T. A. Continuous implicit hybrid one-step methods for the solution of initial value problems of general second-order ordinary differential equations.(Unpublished doctoral dissertation). Covenant University. 2011.

[23] Kuboye, J. O. Block methods for direct solution of higher order ordinary differential equations using interpolation and collocation approach. (Unpublished doctoral thesis). Universiti Utara Malaysia. 2015.

[24] Mohammed, U., Tech, M., Jiya, M., Mohammed, A. A., & Tech, M. A Class of Six Step Block Method for Solution of General Second Order Ordinary Differential Equations. Pacific Journal of Science and Technology. 2010, 11(2), 273 – 277.

November-December 2019
ISSN: 0193-4120 Page No. 764 - 782

Appendix A

$D_{15} = \frac{1}{840pqr}(21r - 280p^2qr - 10 - 4p^6 + 7p^5(3+q+r) - 7q(-3+8r) + 21p - 14p^4(2+3r+q(3+r)) + 35p^3(2r+q(2+3r)) - 56pr + 7pq(-8+35r))$

$D_{25} = -\frac{1}{840pr}(4p^6 + 280p^2qr - 7p^5(3+q+r) + 14p^4(2+3r+q(3+r)) + q^3(-4q^3 + 70r + 7q^2(3+r) - 14q(2+3r)) - 35p^3(2r+q(2+3r)) + 7pq^2(q^3 - 40r - 2q^2(3+r) + 5q(2+3r)))$

$D_{35} = -\frac{1}{840pq}(4p^6 + 280p^2qr - 7p^5(3+q+r) + 14p^4(2+3r+q(3+r)) + 7q(10-6r+r^2) - 35p^3(2r+q(2+3r)) + r^3(r(-28+21r-4r^2)) + r(10-6r+r^2) + 7pr^2(q(15r-2r^2-40)))$

$D_{45} = \frac{1}{420pqr}(32 - 4p^6 - 56qr - 280p^2qr + 7p^5(3+q+r) - 14p^4(2+3r+q(3+r)) + 35p^3(2r+q(2+3r)) + 56p(-r+q(-1+5r)))$

$D_{55} = -\frac{p}{840hqr}(4p^4 + 280qr - 7p^3(3+q+r) + 14p^2(2+3r+q(3+r)) - 70pr - 35pq(2+3r))$

$E_{11} = -\frac{1}{420(-2+p)p(p-q)(p-r)}(10 + 10p^5 - 21r - 2p^4(16+7q+7r) + 7q(8r-3) + p^3(10+49r+7q(7+3r)) - p^2(21r-10+21q(1+4r)) + p(10-21r+56qr-21q))$

$E_{12} = \frac{1}{420(-1+q)(-1+r)}(18 + 4p^5 - 28r + 7q(7r-4) - p^4(10+7q+7r) + p(21r - 10 - 7q(-3+8r)) + p^3(21r-10+7q(3+2r)) + p^2(21r-10-7q(8r-3)))$

$E_{13} = \frac{(-10 - 4p^6 + 21r + 70p^3r + 7p^5(3+r) - 14p^4(2+3r) - 7p(-3+8r))}{420(p-q)q(2-3q+q^2)(q-r)}$

$E_{14} = \frac{(10 + 4p^6 - 21q - 70p^3q - 7p^5(3+q) + 14p^4(2+3q) + 7p(-3+8q))}{420r(-p+r)(-q+r)(2-3r+r^2)}$

$E_{15} = \frac{1}{840(-2+p)(-2+q)(-2+r)}(4 - 4p^6 - 7r + 35p^3qr + 7p^5 + 7p^5q + 7p^5r + 7q(2r-1) - 14p^4(q+r+qr) - 7p(1-2r+q(-2+5r)))$

$E_{21} = \frac{q}{420p(2-3p+p^2)(p-r)}(-10p^5 + 2p^4(2q+7(3+r)) + p^3(4q^2 - 7q(3+r) - 21(2+3r)) + pq(4q^3 - 70r - 7q^2(3+r) + 14q(2+3r)) + q^2(4q^3 - 70r -$

$7q^2(3+r)+14q(2+3r))+p^2(4q^3+70r-7q^2(3+r)+14q(2+3r)))$

$E_{22}=\frac{q}{420(-1+p)(-1+q)(-1+r)}(4p^6-70p^3qr-7p^5(2+q+r)+7pq^3(q^2+10r-2q(2+r))+14p^4(2r+q(2+r))+q^4(-4q^2-28r+7q(2+r)))$

$E_{23}=\frac{1}{420(2-3q+q^2)(q-r)}(-4p^5+p^4(-4q+7(3+r))+p^3(-4q^2+7q(3+r)-14(2+3r))+p^2(-4q^3+70r+7q^2(3+r)-14q(2+3r))+pq(-4q^3+70r+7q^2(3+r)-14q(2+3r))+q^2(10q^3-70r-14q^2(3+r)+21q(2+3r)))$

$E_{24}=\frac{q}{420r(-p+r)(-q+r)(2-3r+r^2)}(4p^6-70p^3q-7p^5(3+q)+14p^4(2+3q)+q^4(21q-28-4q^2)+7pq^3(10-6q+q^2))$

$E_{25}=\frac{q}{840(-2+p)(-2+q)(-2+r)}(-4p^6+35p^3qr+7p^5(1+q+r)-14p^4(q+r+qr)+q^4(4q^2+14r-7q(1+r))-7pq^3(q^2+5r-2q(1+r)))$

$E_{31}=\frac{r}{420p(2-3p+p^2)(p-q)}(-10p^5+2p^4(21+7q+2r)-p^3(42+21r-4r^2+7q(9+r))+p^2(r(28-21r+4r^2))-7q(r^2-10-6r)+pr(r(28-21r+4r^2)-7q(10-6r+r^2))+r^2(-7q(10-6r+r^2)+r(28-21r+4r^2)))$

$E_{32}=\frac{r}{420(-1+p)(-1+q)(-1+r)}(4p^6-70p^3qr-7p^5(2+q+r)+r^4(7q(r-4)+2(7-2r)r)+7pr^3(-2q(-5+r)+(-4+r)r)+14p^4(2r+q(2+r)))$

$E_{33}=\frac{r}{420(p-q)q(2-3q+q^2)(q-r)}(70p^3r-4p^6+7p^5(3+r)-14p^4(2+3r)+r^4(28-21r+4r^2)-7pr^3(10-6r+r^2))$

$E_{34}=\frac{1}{420(q-r)(2-3r+r^2)}(4p^5+p^4(-21-7q+4r)+p^3(28-7q(-6+r)-21r+4r^2)+p^2(-7q(10-6r+r^2)+r(28-21r+4r^2))+pr(-7q(10-6r+r^2)+r(28-21r+4r^2))+r^2(7q(10-9r+2r^2)-2r(21-21r+5r^2)))$

$E_{35}=\frac{r}{840(-2+p)(-2+q)(-2+r)}(-4p^6+35p^3qr+7p^5(1+q+r)-14p^4(q+r+qr)-7pr^3(q(5-2r)+(-2+r)r)+r^4(-7q(-2+r)+r(-7+4r)))$

$E_{41}=\frac{1}{210(-1+p)p(p-q)(p-r)}(16-10p^5-28qr+2p^4(11+7q+7r)+p(8-14qr)+p^2(4+63qr)-p^3(-2+35r+7q(5+3r)))$

November-December 2019
ISSN: 0193-4120 Page No. 764 - 782

$$E_{42} = \frac{1}{210(-1+p)\,(-1+q)\,(-1+r)}\,(4p^6 - 32\,(8+7q\,(-1+r)-7r) - 70p^3qr - 7p^5\,(2+q+r) + 14p^4(2r+q(2+r)) + 56p(4-4r+q(-4+5r)))$$

$$E_{43} = \frac{4}{210(p-q)q(2-3q+q^2)(q-r)}(8 - p^6 - 14pr + \frac{35}{2}p^3r + \frac{7}{4}p^5(3+r) - \frac{14}{4}p^4(2+3r))$$

$$E_{44} = \frac{4}{210r(r-p)(r-q)(2-3r+r^2)}(p^6 - 8 + 14pq - \frac{35}{2}p^3q - \frac{7}{4}p^5(3+q) + \frac{14}{4}p^4(2+3q))$$

$$E_{45} = -\frac{1}{420(-2+q)(-2+r)}(-96 + 4p^5 + p^4(1-7q-7r) - 28q(r-2) + 56r + p(8 - 14qr) + p^2(4-7qr) + 2p^3(1+7qr))$$

$$E_{51} = -\frac{p}{420h(2-3p+p^2)(p-q)(p-r)}\,(10p^4 + 140qr - 14p^3(3+q+r) + 21p^2(2+3r+q(3+r)) - 35p(2r+q(2+3r)))$$

$$E_{52} = \frac{p^3}{420h(-1+p)(-1+q)(-1+r)}(4p^3 - 70qr - 7p^2(2+q+r) + 14p(2r+q(2+r)))$$

$$E_{53} = -\frac{p^3(4p^3-70r-7p^2(3+r)+14p(2+3r))}{420h(p-q)q(2-3q+q^2)(q-r)}, E_{54} = \frac{p^3(4p^3-70q-7p^2(3+q)+14p(2+3q))}{420hr(-p+r)(-q+r)(2-3r+r^2)}$$

$$E_{54} = \frac{p^3(4p^3-70q-7p^2(3+q)+14p(2+3q))}{420hr(-p+r)(-q+r)(2-3r+r^2)}, E_{55} = -\frac{p^3(4p^3-35qr-7p^2(1+q+r)+14p(q+r+qr))}{840h(-2+p)(-2+q)(-2+r)}$$

November-December 2019
ISSN: 0193-4120 Page No. 764 - 782

Appendix B

$$\overline{D}_p = \frac{p^2(4p^4+280qr-7p^3(3+q+r)+14p^2(2+3r+q(3+r))-35p(2r+q(2+3r)))}{840qr}$$

$$\overline{D}_1 = \frac{(-10+21r-7q(-3+8r)+7p(3-8r+q(-8+35r)))}{840pqr}$$

$$\overline{D}_q = \frac{q^2(-7p(q^3-40r-2q^2(3+r)+5q(2+3r))+q(4q^3-70r-7q^2(3+r)+14q(2+3r)))}{840pr}$$

$$\overline{D}_r = \frac{r^2(7p(-r(10-6r+r^2)+q(40-15r+2r^2))+r(-7q(10-6r+r^2)+r(28-21r+4r^2)))}{840pq}$$

$$\overline{D}_2 = \frac{2(4-7qr+7p(-r+q(-1+5r)))}{105pqr}$$

$$\bar{E}_{p1} = \frac{p^2(10p^4+140qr-14p^3(3+q+r)+21p^2(2+3r+q(3+r))-35p(2r+q(2+3r)))}{420(2-3p+p^2)(p-q)(p-r)}$$

$$\bar{E}_{p2} = -\frac{p^4(4p^3-70qr-7p^2(2+q+r)+14p(2r+q(2+r)))}{420(-1+p)(-1+q)(-1+r)}$$

$$\bar{E}_{p3} = \frac{p^4(4p^3-70r-7p^2(3+r)+14p(2+3r))}{420(p-q)q(2-3q+q^2)(q-r)}, \bar{E}_{p4} = -\frac{p^4(4p^3-70q-7p^2(3+q)+14p(2+3q))}{420r(-p+r)(-q+r)(2-3r+r^2)}$$

$$\bar{E}_{p5} = \frac{p^4(4p^3-35qr-7p^2(1+q+r)+14p(q+r+qr))}{840(-2+p)(-2+q)(-2+r)}, \bar{E}_1 = \frac{(10-21r+7q(-3+8r))}{420p(2-3p+p^2)(p-q)(p-r)}$$

$$\bar{E}_2 = \frac{(-18+28r-7q(-4+7r)+7p(4-7r+q(-7+15r)))}{420(-1+p)(-1+q)(-1+r)}, \bar{E}_3 = \frac{(-10+21r-7p(-3+8r))}{420(p-q)q(2-3q+q^2)(q-r)}$$

$$\bar{E}_4 = \frac{(10-21q+7p(-3+8q))}{420r(-p+r)(-q+r)(2-3r+r^2)}, \bar{E}_5 = \frac{(4-7r+7q(-1+2r)-7p(1-2r+q(-2+5r)))}{840(-2+p)(-2+q)(-2+r)}$$

$$\bar{E}_{q1} = \frac{q^4(-4q^3+70r+7q^2(3+r)-14q(2+3r))}{420p(2-3p+p^2)(p-q)(p-r)}, \bar{E}_{q2} = \frac{q^4(7p(q^2+10r-2q(2+r))+q(-4q^2-28r+7q(2+r)))}{420(-1+p)(-1+q)(-1+r)}$$

$$\bar{E}_{q3} = \frac{q^2(q(-10q^3+70r+14q^2(3+r)-21q(2+3r))+7p(2q^3-20r-3q^2(3+r)+5q(2+3r)))}{420(p-q)(2-3q+q^2)(q-r)}$$

$$\bar{E}_{q4} = \frac{q^4(q(-28+21q-4q^2)+7p(10-6q+q^2))}{420r(-p+r)(-q+r)(2-3r+r^2)}, \bar{E}_{q5} = \frac{q^4(q(4q^2+14r-7q(1+r))-7p(q^2+5r-2q(1+r)))}{840(-2+p)(-2+q)(-2+r)}$$

$$\bar{E}_{r1} = \frac{r^4}{420p(2-3p+p^2)(p-q)(p-r)}(r(-28+21r-4r^2)+7q(10-6r+r^2))$$

$$\bar{E}_{r2}=\frac{r^4(r(7q(r-4)+2(7-2r)r)-7p(2q(r-5)-(r-4)r))}{420(p-1)(q-1)(r-1)}$$
$$\bar{E}_{r3}=\frac{r^4}{420(p-q)q(2-3q+q^2)(q-r)}(-7p(10-6r+r^2)+r(28-21r+4r^2))$$
$$\bar{E}_{r4}=\frac{7pr^2(9r^2-10r-2r^3+20q-15qr+3qr^2)+r(2r(21-21r+5r^2)-7q(10-9r+2r^2))}{420(p-r)(q-r)(2-3r+r^2)}$$
$$\bar{E}_{r5}=\frac{r^4}{840(-2+p)(-2+q)(r-2)}(-7p(q(5-2r)+(r-2)r)+r(r(4r-7)-7q(r-2)))$$
$$\bar{E}_2=\frac{4}{105p(2-3p+p^2)(p-q)(p-r)}(-4+7qr),\bar{E}_4=\frac{4(-4(8+7q(-1+r)-7r)+7p(4-4r+q(-4+5r)))}{105(-1+p)(-1+q)(-1+r)}$$
$$\bar{E}_6=\frac{4}{105(p-q)q(2-3q+q^2)(q-r)}(4-7pr),\bar{E}_8=\frac{4}{105r(-p+r)(-q+r)(2-3r+r^2)}(-4+7pq)$$
$$\bar{E}_{10}=-\frac{2}{105(-2+p)(-2+q)(-2+r)}(24+7q(-2+r)-14r+7p(-2+q+r))$$

Appendix C

$$\acute{D}_p=\frac{p(2p^4+60qr-3p^3(3+q+r)+5p^2(2+3r+q(3+r))-10p(2r+q(2+3r)))}{120qr},$$
$$\acute{D}_1=\frac{(-4+q(7-15r)+7r+p(7-15r+5q(-3+10r)))}{120pqr},$$
$$\acute{D}_q=\frac{q(p(q(-20-3(-5+q)q)+5(12+(-6+q)q)r)+q((-2+q)q(-5+2q)+(-20-3(-5+q)q)r))}{120pr},$$
$$\acute{D}_r=\frac{r(60pq-10(2q+p(2+3q))r+5(2+3q+p(3+q))r^2-3(3+p+q)r^3+2r^4)}{120pq},$$
$$\acute{D}_2=\frac{(-2(-2+q+r)+p(-2+5qr))}{15pqr},$$
$$\acute{E}_{p1}=\frac{p(10p^4+60qr-12p^3(3+q+r)+15p^2(2+3r+q(3+r))-20p(2r+q(2+3r)))}{60(2-3p+p^2)(p-q)(p-r)},$$
$$\acute{E}_{p2}=\frac{p^3(20qr-2p^3+3p^2(2+q+r)-5p(2r+q(2+r)))}{60(p-1)(q-1)(r-1)},$$
$$\acute{E}_{p3}=\frac{p^3(2p^3-20r-3p^2(3+r)+5p(2+3r))}{60(p-q)q(2-3q+q^2)(q-r)},\acute{E}_{p4}=\frac{p^3(-2p^3+20q+3p^2(3+q)-5p(2+3q))}{60r(-p+r)(-q+r)(2-3r+r^2)},$$
$$\acute{E}_{p5}=\frac{p^3(2p^3-10qr-3p^2(1+q+r)+5p(q+r+qr))}{120(-2+p)(-2+q)(-2+r)},\acute{E}_1=\frac{(4-7q-7r+15qr)}{60p(2-3p+p^2)(p-q)(p-r)},$$
$$\acute{E}_2=\frac{(-14+18q+18r-25qr+p(18-25r+5q(-5+8r)))}{60(-1+p)(-1+q)(-1+r)},\acute{E}_3=\frac{(-4+p(7-15r)+7r)}{60(p-q)q(2-3q+q^2)(q-r)},$$
$$\acute{E}_4=\frac{(4-7p-7q+15pq)}{60r(-p+r)(-q+r)(2-3r+r^2)},\acute{E}_5=\frac{(2-3r+q(-3+5r)+p(-3+q(5-10r)+5r))}{120(-2+p)(-2+q)(-2+r)},$$
$$\acute{E}_{q1}=\frac{q^3(20r-2q^3+3q^2(3+r)-5q(2+3r))}{60p(2-3p+p^2)(p-q)(p-r)},\acute{E}_{q2}=\frac{q^3(p(3q^2+20r-5q(2+r))+q(q(6-2q+3r)-10r))}{60(-1+p)(-1+q)(-1+r)},$$

$$\acute{E}_{q3}=\frac{q(p(q(40+3q(-15+4q))-15(-2+q)^2r)+q(40r+q(-15(2+3r)+2q(-5q+6(3+r)))))}{60(p-q)(-2+q)(-1+q)(q-r)},$$
$$\acute{E}_{q4}=\frac{q^3((2-q)q(2q-5)+p(20+3(q-5)q))}{60(-2+r)(-1+r)r(-p+r)(-q+r)},\acute{E}_{q5}=\frac{q^3(p((5-3q)q+5(q-2)r)+q(2q^2+5r-3q(1+r)))}{120(-2+p)(-2+q)(-2+r)},$$
$$\acute{E}_{r1}=\frac{r^3((2-r)r(-5+2r)+q(20+3(-5+r)r))}{60(-2+p)(-1+p)p(p-q)(p-r)},\acute{E}_{r2}=\frac{r^3(20pq-5(2q+p(2+q))r+3(2+p+q)r^2-2r^3)}{60(-1+p)(-1+q)(-1+r)},$$
$$\acute{E}_{r3}=\frac{r^3((-2+r)r(-5+2r)+p(-20-3(-5+r)r))}{60(p-q)(-2+q)(-1+q)q(q-r)},$$
$$\acute{E}_{r4}=\frac{r(60pq-20(2q+p(2+3q))r+15(2+3q+p(3+q))r^2-12(3+p+q)r^3+10r^4)}{60(p-r)(q-r)(-2+r)(-1+r)},$$
$$\acute{E}_{r5}=\frac{r^3(-10pq+5(p+q+pq)r-3(1+p+q)r^2+2r^3)}{120(-2+p)(-2+q)(-2+r)},\acute{E}_2=\frac{4(-2+q+r)}{15p(2-3p+p^2)(p-q)(p-r)},$$
$$\acute{E}_4=\frac{4(-8+q(6-5r)+p(6+5q(-1+r)-5r)+6r)}{15(-1+p)(-1+q)(-1+r)},\acute{E}_6=-\frac{4(-2+p+r)}{15(p-q)q(2-3q+q^2)(q-r)},$$
$$\acute{E}_8=\frac{4(-2+p+q)}{15r(-p+r)(-q+r)(2-3r+r^2)},\acute{E}_{10}=\frac{(p(18+5q(-2+r)-10r)+2(-16+q(9-5r)+9r))}{15(-2+p)(-2+q)(-2+r)}.$$